Firefighting on Geometric Graphs with Density Bounds

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Abstract

Let G be an infinite geometric graph; in particular, a graph whose vertices are a countable discrete set of points on the plane, with vertices u, v adjacent if their Euclidean distance is less than 1. A "fire" begins at some finite set of vertices and spreads to all neighbors in discrete steps; in the meantime f vertices can be deleted at each time-step. Let $f(G)$ be the least f for which any fire on G can be stopped in finite time. We show that if G has bounded density, in the sense that no open disk of radius r contains more than λ vertices, then $f(G)$ is bounded above by the ceiling of a universal constant times λ/r^2. Similarly, if the density of G is bounded from below in the sense that every open disk of radius r contains at least κ vertices, then $f(G)$ is bounded below by κ times the square of the floor of a universal constant times $1/r$.

Keywords: firefighting, geometric graph.

1 Introduction

The Firefighter Problem was introduced by Hartnell [4] in 1995. A fire starts at a vertex in a graph G at time 0, and spreads to all neighboring vertices in successive discrete time steps. Between each of these epochs, f vertices are "protected" (equivalently, removed), where f is some fixed positive integer representing the number of firefighters. When a vertex is burning or has been protected, it remains in that state. The process terminates when the fire can not spread any longer; the objective, when G is infinite, is to determine the minimum number of firefighters needed to stop any fire in finite time.

For a survey of results, see [2]. So far, most of the work done on infinite graphs has been on plane lattices such as the square, triangular, and hexagonal lattice [3, 6, 7]. For fighting fires in a forest, as opposed to an orchard, it is natural instead to consider graphs whose vertices are distributed on the Euclidean plane according to some density conditions. In the first model, due to limitations on space and resources, we assume that there is an upper bound on the number of vertices that are in any disk of fixed radius. In the second model, we are interested in a forest that is not too thin, and consequently, we assume that there is a lower bound on the number of vertices that are in any disk of fixed radius. Moreover, in both models, we are assuming that vertices are adjacent when they are close enough to permit the fire to spread from one vertex to another, giving rise to geometric graphs studied here.

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2 Preliminaries and Main Results

Definition 2.1. *A graph G is* geometric *if its vertices are points of the plane $\mathbb{R}^2$, with u, v adjacent just when their Euclidean distance $\rho(u, v)$ is less than one.*

Let G be a fixed geometric graph. If S is a finite set of vertices in G, we denote by $f_S(G)$ the minimum number of firefighters needed to stop a fire that begins at (or has expanded to, by the time firefighting commences) the vertices in S. Provided that it exists, let $f(G)$ be the maximum of $f_S(G)$ over all finite vertex sets S, i.e.,

$$f(G) = \max_{S \subset V(G), |S| < \infty} f_G(S).$$

Note that if a graph G has no infinite component, then no firefighters are needed, hence $f(G) = 0$.

Definition 2.2. *Let r be a fixed positive real number and λ a fixed positive integer. Let $\mathfrak{G}^{r,\lambda}$ be the set of all geometric graphs G in which every open ball of radius r contains at most λ vertices.*

As we shall see, for any $G \in \mathfrak{G}^{r,\lambda}$, $f(G)$ exists, and, in fact, there is a finite value, denoted by $f^{r,\lambda}$, representing the smallest number of firefighters needed in order to stop any fire in any $G \in \mathfrak{G}^{r,\lambda}$. In other words,

$$f^{r,\lambda} = \max_{G \in \mathfrak{G}^{r,\lambda}} f(G).$$

Theorem 2.3. *There exists a constant C such that for all r and λ, $f^{r,\lambda} \leq \left\lceil \frac{C\lambda}{r^2} \right\rceil$.*

Theorem 2.4. *There exists a constant $C' > 0$ such that for all λ, $f^{r,\lambda} \geq \left\lfloor \frac{C'\lambda}{r^2} \right\rfloor$.*

Definition 2.5. *Let r be a fixed positive real number and κ a fixed positive integer. Let $\mathfrak{G}_{r,\kappa}$ be the set of all geometric graphs G in which every open ball of radius r contains at least κ vertices.*

In the case of lower bounded geometric graphs, we define

$$f_{r,\kappa} = \min_{G \in \mathfrak{G}_{r,\kappa}} f(G)$$

so that $f_{r,\kappa}$ is the minumum, over all $G \in \mathfrak{G}_{r,\kappa}$, of the number of firefighters needed to stop any fire in G.

Theorem 2.6. *There exists a constant C such that for all r and κ, $f_{r,\kappa} \geq \left\lfloor \frac{C}{r} \right\rfloor^2 \kappa$.*

3 Proof of Main Results

3.1 Upper Bounded Density

We will prove Theorem 2.3 via Propositions 3.1 and 3.2. Let $\mathcal{C}_r(x)$, $\mathcal{B}_r(x)$, and $\mathcal{D}_r(x)$ denote, respectively, the circle, open disk, and closed disk of radius r and center x. When x is the origin, we will simply denote them by $\mathcal{C}_r$, $\mathcal{B}_r$, and $\mathcal{D}_r$, unless stated otherwise.

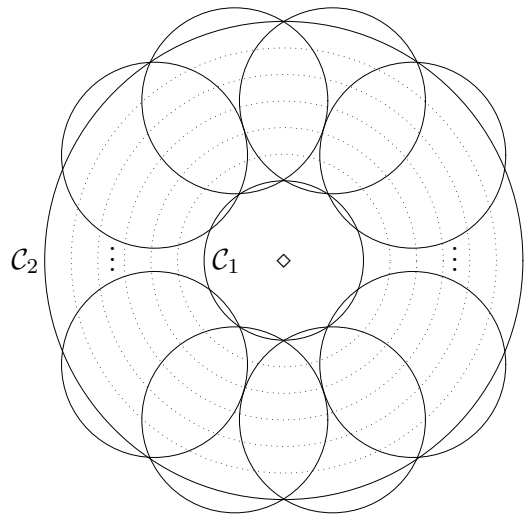

Figure 1: An Efficient Covering of $\mathcal{A}$ with Open Balls of Radius r

Proposition 3.1. *When $r \geq 1$,*

$$f^{r,\lambda} \leq \Big\lceil \frac{4\pi\lambda}{(2-\nu)r^2} \Big\rceil,$$

where $\nu = \frac{\sqrt{2}}{2} + \frac{1}{2}$.

Proof. Suppose $G \in \mathfrak{G}^{r,\lambda}$ and let us assume that the fire has started in a finite set of vertices S of G. Let $\mathcal{D}$ be the smallest closed disk that contains all the vertices of S. For simplicity of notation, we assume that the center of this disk is the origin $\mathbf{0}$ and denote its radius by d. We will try to find circles $\mathcal{C}_1 : x^2 + y^2 = R^2$ and $\mathcal{C}_2 : x^2 + y^2 = (R+m)^2$ for a large integer R and positive integer $0 \leq l < m < 2r$ such that the firefighters can defend all the vertices in the annulus $\mathcal{A}_l : (R+l)^2 \leq x^2 + y^2 < (R+l+1)^2$ by the time the fire has reached $\mathcal{C}_1$. In order to find such an annulus, we will cover $\mathcal{A} : R^2 \leq x^2 + y^2 < (R+m)^2$, the annulus between $\mathcal{C}_1$ and $\mathcal{C}_2$, with open balls of radius r efficiently (see Figure 1). Suppose n open balls are needed to cover $\mathcal{A}$. Since each open ball contains at most λ vertices, there are at most $n\lambda$ vertices in $\mathcal{A}$. Since $\mathcal{A} = \bigcup_{0 \leq l < m} \mathcal{A}_l$, we know that there exists l_0 for which $\mathcal{A}_{l_0}$ contains at most $\frac{n\lambda}{m}$ vertices. Since the fire cannot spread radially at rate greater than 1, it will reach $\mathcal{C}_1$ after $R - \lceil d \rceil$ time units. As a result, we need at most $\frac{n\lambda}{m(R-\lceil d \rceil)}$ firefighters per turn to defend all the vertices in $\mathcal{A}_{l_0}$. Since this annulus is of annular width one, the fire will not spread beyond the inner circle of $\mathcal{A}_{l_0}$; therefore, $f_S(G) \leq \lceil \frac{n\lambda}{m(R-\lceil d \rceil)} \rceil$.

Let $\mathcal{B}$ be a disk of radius r whose center is on the circle $\mathcal{C}_s : x^2 + y^2 = (R+sm)^2$ for some $0 < s < 1$. We assume that $\mathcal{B}$ intersects $\mathcal{C}_1$ at distinct points p_1 and q_1 and $\mathcal{C}_2$ at distinct corresponding points p_2 and q_2. Our objective is to make p_1, p_2, and $\mathbf{0}$ (similarly, q_1, q_2, and $\mathbf{0}$) collinear by adjusting s. For simplicity of computations, let us assume that the center of $\mathcal{B}$ is the point $(0, R+sm)$. In order to find $p_i = (x_i, y_i)$, we need to solve the following system of equations:

$$x_1^2 + y_1^2 = R^2,\ x_1^2 + (y_1 - (R+sm))^2 = r^2,\ y_1 = tx_1,$$
$$x_2^2 + y_2^2 = (R+m)^2,\ x_2^2 + (y_2 - (R+sm))^2 = r^2,\ y_2 = tx_2.$$

By doing so, we have

$$(x_1, y_1) = \Big(\frac{R}{2} \sqrt{\frac{4r^2 - m^2}{R^2 + mR + r^2}}, \frac{2R^2 + mR}{\sqrt{R^2 + mR + r^2}} \Big).$$

Since p_1, p_2, and $\mathbf{0}$ are collinear, we can cover $\mathcal{A}$ with $\lceil \frac{2\pi}{\theta} \rceil$ copies of $\mathcal{B}$ where θ is the central angle of the sector of $\mathcal{C}_1$ that is inside $\mathcal{B}$. We know that $\theta = 2\arcsin(\frac{x_1}{R})$. Having the Taylor expansion of arcsin in mind, we have

$$\theta \geq \sqrt{\frac{4r^2 - m^2}{R^2 + mR + r^2}},$$

and, as a result, we can cover $\mathcal{A}$ with at most

$$\Big\lceil 2\pi \sqrt{\frac{R^2 + mR + r^2}{4r^2 - m^2}} \Big\rceil$$

copies of $\mathcal{B}$. Since n is the least number of open balls of radius r to cover $\mathcal{A}$, we have

$$n \leq \Big\lceil 2\pi \sqrt{\frac{R^2 + mR + r^2}{4r^2 - m^2}} \Big\rceil.$$

Returning to our earlier argument, we have

$$f_S(G) \leq \Big\lceil \frac{n\lambda}{m(R - \lceil d \rceil)} \Big\rceil \leq \Big\lceil \Big\lceil 2\pi \sqrt{\frac{R^2 + mR + r^2}{4r^2 - m^2}} \Big\rceil \frac{\lambda}{m(R - \lceil d \rceil)} \Big\rceil.$$

It follows that

$$\Big| \Big\lceil 2\pi \sqrt{\frac{R^2 + mR + r^2}{4r^2 - m^2}} \Big\rceil \frac{1}{R - \lceil d \rceil} - 2\pi \sqrt{\frac{R^2 + mR + r^2}{4r^2 - m^2}} \frac{1}{R - \lceil d \rceil} \Big| =$$

$$\Big| \frac{1}{R - \lceil d \rceil} \Big(1 - \langle 2\pi \sqrt{\frac{R^2 + mR + r^2}{4r^2 - m^2}} \rangle \Big) \Big| \leq \frac{1}{R - \lceil d \rceil},$$

where $\langle x \rangle$ is the fractional part of x. As a result, we have

$$f_S(G) \leq \Big\lceil \frac{\sqrt{R^2 + mR + r^2} + \sqrt{4r^2 - m^2}}{R - \lceil d \rceil} \cdot \frac{2\pi\lambda}{m\sqrt{4r^2 - m^2}} \Big\rceil.$$

Suppose η denotes $\sqrt{4r^2 - m^2}$ and we have

$$\Big(\sqrt{R^2 + mR + r^2} + \eta \Big) < \Big(\sqrt{R^2 + 2mR + r^2} + \eta \Big) = \Big(\sqrt{(R+m)^2 + (r^2 - m^2)} + \eta \Big)$$

$$< \Big(\sqrt{(R+m)^2 + (4r^2 - m^2)} + \eta \Big) \leq \Big(\sqrt{(R+m)^2} + \sqrt{4r^2 - m^2} + \eta \Big) = (R + m) + 2\eta.$$

Given that R is greater than $m + 2\eta + 2\lceil d \rceil$, we have

$$\frac{\sqrt{R^2 + mR + r^2} + \sqrt{4r^2 - m^2}}{R - \lceil d \rceil} < \frac{R + m + 2\eta}{R - \lceil d \rceil} < 2.$$

It follows that

$$f_S(G) \leq \Big\lceil \frac{4\pi\lambda}{m\sqrt{4r^2 - m^2}} \Big\rceil.$$

For $m_0 = \sqrt{2}r$, $m\sqrt{4r^2 - m^2}$ is minimized for real values of m with the minimum value $2r^2$. To make m an integer, we will use the closest integer to $m_0 = \sqrt{2}r$ for this purpose. We now compute the error bounds that arise from doing so. We may assume that m_0 is not an

integer. Two cases arise: $\frac{1}{2} > \langle m_0 \rangle$ and $1 > \langle m_0 \rangle \geq \frac{1}{2}$. We know that $\lfloor x \rfloor = x - \langle x \rangle$ and when x is not an integer, $\lceil x \rceil = \lfloor x \rfloor + 1$.

When $\frac{1}{2} > \langle m_0 \rangle$, the closest integer to $m_0 = \sqrt{2}r$ is its floor, so we want to find an upper and a lower bound for

$$\lfloor \sqrt{2}r \rfloor \Big(\sqrt{4r^2 - \lfloor \sqrt{2}r \rfloor^2} \Big) - 2r^2 =$$

$$\lfloor \sqrt{2}r \rfloor \Big(\sqrt{2r^2 + (\sqrt{2}r - \langle \sqrt{2}r \rangle)\langle \sqrt{2}r \rangle} \Big) - 2r^2 =$$

$$\lfloor \sqrt{2}r \rfloor \Big(\sqrt{(\sqrt{2}r + \frac{\langle \sqrt{2}r \rangle}{2})^2 - \frac{3}{4}\langle \sqrt{2}r \rangle^2} \Big) - 2r^2.$$

On one hand, we have $\sqrt{2}r - \frac{1}{2} < \lfloor \sqrt{2}r \rfloor < \sqrt{2}r$. On the other hand, we have the following upper bound for the expression under the radical:

$$\sqrt{(\sqrt{2}r + \frac{\langle \sqrt{2}r \rangle}{2})^2 - \frac{3}{4}\langle \sqrt{2}r \rangle^2} < \sqrt{2}r + \frac{\langle \sqrt{2}r \rangle}{2} < \sqrt{2}r + \frac{1}{4}.$$

Since $\langle \sqrt{2}r \rangle < 1$, it follows that $\langle \sqrt{2}r \rangle > \langle \sqrt{2}r \rangle^2$. By assumption, $r \geq 1$ which implies that $\sqrt{2}r > 1$. Putting these two observations together, we conclude that $(\sqrt{2}r - \langle \sqrt{2}r \rangle)\langle \sqrt{2}r \rangle > 0$. As a result, we have the following lower bound for the expression under the radical:

$$\sqrt{2}r < \sqrt{2r^2 + (\sqrt{2}r - \langle \sqrt{2}r \rangle)\langle \sqrt{2}r \rangle}.$$

We conclude that

$$2r^2 - \frac{\sqrt{2}}{2}r < \lfloor \sqrt{2}r \rfloor \sqrt{4r^2 - (\sqrt{2}r - \langle \sqrt{2}r \rangle)^2} < 2r^2 + \frac{\sqrt{2}}{4}r$$

which implies that

$$\Big| \lfloor \sqrt{2}r \rfloor (\sqrt{4r^2 - \lfloor \sqrt{2}r \rfloor^2}) - 2r^2 \Big| < \frac{\sqrt{2}}{2}r.$$

Now assume that $\frac{1}{2} \leq \langle m_0 \rangle < 1$. In this case, the closest integer to $m_0 = \sqrt{2}r$ is its ceiling, and, as before, we want to find a lower and an upper bound for

$$\lceil \sqrt{2}r \rceil \Big(\sqrt{4r^2 - \lceil \sqrt{2}r \rceil^2} \Big) - 2r^2 =$$

$$\lceil \sqrt{2}r \rceil \Big(\sqrt{2r^2 - 2\sqrt{2}r(1 - \langle \sqrt{2}r \rangle) - (1 - \langle \sqrt{2}r \rangle)^2} \Big) - 2r^2.$$

On one hand, since $0 < 1 - \langle \sqrt{2}r \rangle \leq \frac{1}{2}$, we have $\sqrt{2}r < \lceil \sqrt{2}r \rceil < \sqrt{2}r + \frac{1}{2}$. On the other hand, since $0 < 2\sqrt{2}r(1 - \langle \sqrt{2}r \rangle) \leq \sqrt{2}r$ and $0 < (1 - \langle \sqrt{2}r \rangle)^2 \leq \frac{1}{4}$, we have the following bounds for the expression under the radical:

$$\sqrt{2r^2 - \sqrt{2}r - \frac{1}{4}} \leq \sqrt{2r^2 - 2\sqrt{2}r(1 - \langle \sqrt{2}r \rangle) - (1 - \langle \sqrt{2}r \rangle)^2} < \sqrt{2}r.$$

The expression on the left can be rewritten as $\sqrt{(\sqrt{2}r - \frac{1}{2})^2 - \frac{1}{8}}$. Since $\sqrt{x^2 - y^2} > \sqrt{(x-y)^2}$ when $x > y > 0$, we have

$$\sqrt{2}r - \frac{\sqrt{2}}{4} - \frac{1}{2} < \sqrt{(\sqrt{2}r - \frac{1}{2})^2 - \frac{1}{8}}.$$

Putting these observations together, we have

$$\lceil\sqrt{2}r\rceil\Big(\sqrt{4r^2-(\sqrt{2}r+1-\langle\sqrt{2}r\rangle)^2}\Big) < 2r^2+\frac{\sqrt{2}}{2}r$$

and

$$2r^2-\frac{\sqrt{2}}{2}(1+\frac{\sqrt{2}}{2})r < \lceil\sqrt{2}r\rceil\Big(\sqrt{4r^2-(\sqrt{2}r+1-\langle\sqrt{2}r\rangle)^2}\Big)$$

which implies that

$$\Big|\lceil\sqrt{2}r\rceil(\sqrt{4r^2-\lceil\sqrt{2}r\rceil^2})-2r^2\Big| < \frac{\sqrt{2}}{2}(1+\frac{\sqrt{2}}{2})r.$$

In either cases, the closest integer to m_0 gives us an approximation for $m_0\sqrt{4r^2-m_0}=2r^2$, and we have

$$f_S(G) \le \Big\lceil\frac{4\pi\lambda}{2r^2-\nu r}\Big\rceil \le \Big\lceil\frac{4\pi\lambda}{(2-\nu)r^2}\Big\rceil,$$

where $\nu=\frac{\sqrt{2}}{2}+\frac{1}{2}$. Since the right hand side of the inequality is an upper bound for $f_S(G)$ for any finite subset S of vertices in G where G is an arbitrary member of $\mathfrak{G}^{r,\lambda}$, we have

$$f^{r,\lambda} \le \Big\lceil\frac{4\pi\lambda}{(2-\nu)r^2}\Big\rceil. \qquad \square$$

Proposition 3.2. *When $r<1$,*

$$f^{r,\lambda} \le \Big\lceil\frac{2\pi}{r^2}\lambda\Big\rceil.$$

Proof. Suppose $G\in\mathfrak{G}^{r,\lambda}$ and let us assume that the fire has started in a finite set of vertices S of G. Let $\mathcal{D}$ be the smallest closed disk that contains all the vertices of S. For simplicity of notation, we assume that the center of this disk is the origin $\mathbf{0}$ and denote its radius by d.

Suppose $\mathbb{H}_r$ is the hexagonal lattice whose faces are regular hexagons of side r. Moreover, let us assume that the origin is a vertex of $\mathbb{H}_r$. Suppose $\mathcal{H}$ is an arbitrary closed face of $\mathbb{H}_r$. For a fixed $\theta\in(-\pi,\pi]$, let $\mathfrak{L}_\theta(\mathcal{H})$ be the set of all lines that intersect $\mathcal{H}$ and have $\tan(\theta)$ as their slope. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are two extremal elements of $\mathfrak{L}_\theta(\mathcal{H})$, i.e, with the largest and smallest y-intercept when $\theta\in(-\pi,\pi)$ or the largest and smallest x-intercept when $\theta=\pi$. Then $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect $\mathcal{H}$ at two opposite corners or at two opposite sides of $\mathcal{H}$. Let p and q to be two antipodal points on the boundary of $\mathcal{H}$. Clearly, $\rho(p,q)\le 2r$.

Now we want to demonstrate that when R is large enough, provided that $\mathcal{C}_R\cap\mathcal{H}\neq\emptyset$, the difference between $\rho(p,q)$ and the arc length of the sector they mark on $\mathcal{C}_R$ is negligible. Suppose these two points have a distance of $2r$ which is the maximum distance between any two points on the boundary of $\mathcal{H}$. We know that the arc length between them is equal to $2R\arcsin(\frac{r}{R})$. For $\epsilon>0$, provided that $R>\sqrt{\frac{2}{\epsilon}(\arcsin(r)-r)}$, we have

$$\frac{\epsilon}{2} > \frac{1}{R^2}\Big(\sum_{n=1}^{\infty}\frac{(\frac{1}{2})_n}{(2n+1)n!}r^{2n+1}\Big) \ge R\Big(\sum_{n=1}^{\infty}\frac{(\frac{1}{2})_n}{(2n+1)n!}\Big(\frac{r}{R}\Big)^{2n+1}\Big) = R\arcsin\Big(\frac{r}{R}\Big)-r,$$

where $(x)_n=x(x-1)\cdots(x-n+1)$. Consequently, $\epsilon>2R\arcsin(\frac{r}{R})-2r\ge 0$.

Define $g_1(R)$ and $g_2(R)$ to be number of hexagons intersecting and contained in the interior of $\mathcal{D}_R$, respectively. Let $k(R)$ be the number of faces of $\mathbb{H}_r$ that is in the annulus $\mathcal{A}_R: R^2\le x^2+y^2<(R+1)^2$. We choose $R_0>\lceil d\rceil$ so that $\mathcal{C}_{R_0}$ has negligible curvature relative to the

distance r and choose $R_1 > R_0$ such that the number of hexagons that intersect the (R_0, R_1+1)-annulus $R_0^2 \leq x^2 + y^2 < (R_1+1)^2$ is less than $4\pi((R_1+1)^2 - R_0^2)/(3\sqrt{3}r^2)$. This radius exists because on average an $(R_0, R+1)$-annulus contains $2\pi((R+1)^2 - R_0^2)/(3\sqrt{3}r^2)$ hexagons.

Let us assume that the hexagons covering the (R_0, R_1+1)-annulus are indexed by i. Denoting the characteristic function by χ, we have

$$\int_{R_0}^{R_1} k(R)dR \leq \int_{R_0}^{R_1} \sum_i \chi(\mathcal{A}_R \cap \mathcal{H}_i)dR = \sum_i \int_{R_0}^{R_1} \chi(\mathcal{A}_R \cap \mathcal{H}_i)dR,$$

where $\mathcal{H}_i$ is the i-th hexagon in the aforementioned annulus. But $\int_{R_0}^{R_1} \chi(\mathcal{A}_R \cap \mathcal{H}_i)dR$ is equal to the difference between the maximum and the minimum value of R such that $\mathcal{A}_R$ intersects $\mathcal{H}_i$. In other words, since the annular width of each $\mathcal{A}_R$ is one, this difference is equal to one plus the width of $\mathcal{H}_i$ in the particular direction that circles centered at the origin sweep through $\mathcal{H}_i$. On the other hand, we know that the diameter of each $\mathcal{H}_i$ is $2r$, and, as a result,

$$\int_{R_0}^{R_1} k(R)dR \leq (g_1(R_1+1) - g_2(R_0))(1+2r),$$

where $g_1(R_1+1) - g_2(R_0)$ is number of hexagons covering the (R_0, R_1+1)-annulus. We want $(g_1(R_1+1) - g_2(R_0))(1+2r)$ to be at most

$$\frac{4\pi((R_1+1)^2 - R_0^2)}{3\sqrt{3}r^2};$$

therefore,

$$g_1(R_1+1) - g_2(R_0) \leq \frac{4\pi((R_1+1)^2 - R_0^2)}{3\sqrt{3}r^2(1+2r)}.$$

Let γ be the right hand side of this inequality divided by $(R_1^2 - R_0^2)/2(1+2r)$. We claim that for some $R' \in [R_0, R_1]$, $k(R') \leq \gamma R'$. Assume the contrary, and we have

$$\int_{R_0}^{R_1} k(R)dR > \frac{\gamma R^2}{2}\Big|_{R_0}^{R_1} = \frac{4\pi((R_1+1)^2 - R_0^2)}{3\sqrt{3}r^2},$$

which is a contradiction; hence, there is $R' \in [R_0, R_1]$ such that there are at most

$$\frac{2\pi}{3\sqrt{3}r^2}\Big(\frac{(R_1+1)^2 - R_0^2}{R_1^2 - R_0^2}\Big)R' = \frac{2\pi}{3\sqrt{3}r^2}\Big(1 + \frac{2R_1+1}{R_1^2 - R_0^2}\Big)R'$$

hexagons covering $\mathcal{A}_{R'}$. Given that $R - 1 \geq R_0 \geq 1$,

$$\frac{2R+1}{R^2 - R_0^2}$$

is a strictly decreasing function of R which is bounded from above by 2. Its value at $R = R_0+1$ is equal to $(2R_0+3)/(2R_0+1)$ which is less than 2. Consequently, there are at most

$$\frac{2\pi}{\sqrt{3}r^2}R'$$

hexagons covering $\mathcal{A}_{R'}$.

On the other hand, each hexagon is circumscribed in a closed disk of radius r which contains at most λ vertices. As a result, there are at most

$$\frac{2\pi}{\sqrt{3}r^2}R'\lambda$$

vertices in $\mathcal{A}_{R'}$. Since the fire cannot spread radially at rate greater than 1, we have

$$f_S(G) \leq \frac{2\pi\lambda R'}{\sqrt{3}r^2(\lfloor R' \rfloor - 1 - \lceil d \rceil)}.$$

Provided that $\sqrt{3}(\lceil d \rceil + 1) < (\sqrt{3} - 1)\lfloor R_0 \rfloor$, we have $\frac{R'}{\lfloor R' \rfloor - 1 - \lceil d \rceil} < \sqrt{3}$, and consequently,

$$f_S(G) \leq \left\lceil \frac{2\pi\lambda}{r^2} \right\rceil.$$

Since the right hand side of the inequality is an upper bound for $f_S(G)$ for any finite subset S of vertices in G where G is an arbitrary member of $\mathfrak{G}^{r,\lambda}$, we have

$$f^{r,\lambda} \leq \left\lceil \frac{2\pi\lambda}{r^2} \right\rceil. \qquad \square$$

In order to prove Theorem 2.4, we will prove the following proposition.

Proposition 3.3. *Assuming $\lambda \geq 1$,*

1. $f^{r,\lambda} \geq \lfloor \frac{\lambda}{19r^2} \rfloor$, *for* $1 \leq r$;
2. $f^{r,\lambda} \geq 4\lfloor \frac{\lambda}{76r^2} \rfloor$, *for* $\frac{1}{2} < r < 1$;
3. $f^{r,\lambda} \geq 2\lambda \geq 2\lfloor \frac{\lambda}{2r^2} \rfloor$, *for* $\frac{\sqrt{2}}{2} \leq r \leq \frac{1}{2}$;
4. $f^{r,\lambda} \geq 4\lambda \geq 4\lfloor \frac{2\lambda}{81r^2} \rfloor$, *for* $\frac{\sqrt{2}}{9} \leq r < \frac{\sqrt{2}}{2}$;
5. $f^{r,\lambda} \geq 4\lfloor \frac{\sqrt{2}}{9r} \rfloor^2 \lambda \geq \lfloor \frac{2\lambda}{81r^2} \rfloor$, *for* $0 < r < \frac{\sqrt{2}}{9}$.

In the proof of this proposition, we use the following lemma whose proof can also be found in [1].

Lemma 3.4. *When $r \geq 1$, the number of points in $\mathbb{Z}^2$ inside an open ball of radius r is bounded from above by $\pi r^2 + 4r + 11$.*

Proof. (Proposition 3.3) Let $s \cdot \mathbb{Z}^2$ be the square lattice with side length s. In this proof, by a lattice point we mean a point in $s \cdot \mathbb{Z}^2$, unless stated otherwise.

Let $1 \leq r$. For a sufficiently small positive real number ϵ, consider $(1-\epsilon) \cdot \mathbb{Z}^2$. Since the Euclidean distance between any pair of consecutive horizontal or vertical lattice points in $(1-\epsilon) \cdot \mathbb{Z}^2$ is slightly smaller than 1, as a geometric graph, $\mathbb{Z}^2_\epsilon$ is isomorphic to $\mathbb{Z}^2$. By Lemma 3.4, we also know that any circle of radius $r \geq 1$ contains at most $\pi r^2(1-\epsilon)^{-2} + 4r(1-\epsilon)^{-1} + 11$ points in $\mathbb{Z}^2_\epsilon$.

Centered at each lattice point, consider an open ball of radius $\epsilon/2$, and for a fixed positive integer m, distribute m points in each ball at random. Let the these points to be the vertices of a geometric graph G. For every lattice point p, the vertices in $\mathcal{B}_{\epsilon/2}(p)$ form a clique. Moreover, if p and q are two lattice points that are adjacent in $\mathbb{Z}^2_\epsilon$, then any vertex in $\mathcal{B}_{\epsilon/2}(p)$ is adjacent

to any vertex in $\mathcal{B}_{\epsilon/2}(q)$. Assuming that $\epsilon < 1 - \sqrt{2}/2$, these are the only adjacencies that we have in G. We know that 2 firefighters are necessary and sufficient to stop any finite fire in the square lattice [3]. Considering all the vertices that we distributed around each lattice point as a mega-vertex and the edges that join the vertices in one mega-vertex to another mega-vertex as a mega-edge, we will have a graph isomorphic to $\mathbb{Z}^2$. In order to protect a mega-vertex from catching and spreading the fire, we need to protect all the m vertices it contains. Let $m = \lfloor \lambda/(\pi r^2(1-\epsilon)^{-2} + 4r(1-\epsilon)^{-1} + 11) \rfloor$ and construct G as above for this particular value of m. It follows that each circle of radius r has at most λ vertices. Since $2m$ firefighters are needed to stop the fire, we have

$$f(G) \geq 2\Big\lfloor \frac{\lambda(1-\epsilon)^2}{\pi r^2 + 4r(1-\epsilon) + 11(1-\epsilon)^2} \Big\rfloor \geq 2\Big\lfloor \frac{\lambda(1-\epsilon)^2}{\pi r^2 + 4r + 11} \Big\rfloor.$$

Since $r \geq 1$, assuming that $\epsilon = (0.1)$, we have

$$f^{r,\lambda} \geq 2\Big\lfloor \frac{(.81)\lambda}{\pi r^2 + 4r + 11} \Big\rfloor \geq (1.62)\Big\lfloor \frac{\lambda}{\pi r^2 + 4r + 11} \Big\rfloor \geq \Big\lfloor \frac{\lambda}{\pi r^2 + 4r + 11} \Big\rfloor \geq \Big\lfloor \frac{\lambda}{19r^2} \Big\rfloor.$$

Suppose $\frac{1}{2} < r < 1$ and let $0 < \epsilon < r - \frac{1}{2}$. In this case, consider $(\frac{1}{2} + \epsilon) \cdot \mathbb{Z}^2$. Assuming that a vertex of a graph is present at each lattice point, then $(\frac{1}{2} + \epsilon) \cdot \mathbb{Z}^2$, as a geometric graph, is isomorphic to the strong square lattice $\mathbb{Z}^2_{\boxtimes}$. By Proposition 3.4, we also know that any circle of radius r contains at most $\pi r^2(\frac{1}{2} + \epsilon)^2 + 4r(\frac{1}{2} + \epsilon) + 11$ lattice points in $(\frac{1}{2} + \epsilon) \cdot \mathbb{Z}^2$.

As in the previous case, construct a geometric graph G where centered at each lattice point we consider an open ball of radius $\epsilon/2$ and distribute $\lfloor \lambda/(\pi r^2(\frac{1}{2} + \epsilon)^2 + 4r(\frac{1}{2} + \epsilon) + 11) \rfloor$ vertices in each open ball at random. We know that 4 firefighters are necessary and sufficient to stop any finite fire in the square lattice [6]. Considering all the vertices distributed around each lattice point as a mega-vertex and the edges that join the vertices in one mega-vertex to another mega-vertex as a mega-edge, we will have a graph isomorphic to $\mathbb{Z}^2_{\boxtimes}$. It follows that

$$f(G) \geq 4\Big\lfloor \frac{\lambda}{\frac{\pi}{4} r^2(1+2\epsilon)^2 + 2r(1+2\epsilon) + 11} \Big\rfloor.$$

Assuming that $0 < \epsilon < \frac{1}{2}$, we have $(1 + 2\epsilon) < 2$, and consequently,

$$f^{r,\lambda} \geq 4\Big\lfloor \frac{\lambda}{\pi r^2 + 4r + 11} \Big\rfloor.$$

On one hand, we have $f^{r,\lambda} \geq 4\lfloor \frac{\lambda}{19} \rfloor$ since $r < 1$. On the other hand, since $r^2 > \frac{1}{4}$, we have $\frac{\lambda}{19} > \frac{\lambda}{76r^2}$. It follows that

$$f^{r,\lambda} \geq 4\Big\lfloor \frac{\lambda}{76r^2} \Big\rfloor.$$

Let $\frac{\sqrt{2}}{2} \leq r \leq \frac{1}{2}$ and let $0 < \epsilon < 1 - 2r$. Centered at every lattice point in $(1-\epsilon) \cdot \mathbb{Z}^2$ consider open balls of radius $\epsilon/2$. Denote the set of all these open balls by $\mathfrak{B}$. In each of these open ball, we randomly choose λ distinct points and we let G to be the geometric graph whose vertices are these randomly chosen points for all open balls in $\mathfrak{B}$. We will show that $G \in \mathfrak{G}^{r,\lambda}$. Clearly, every open ball of radius r centered at one the lattice points contains λ vertices. Since the distance between any two elements in $\mathfrak{B}$ is at least $1 - \epsilon$ and the diameter of an open ball of radius r is less than $1 - \epsilon$, any other open ball of radius r can only contain vertices that belong to exactly one of the open balls in $\mathfrak{B}$. Consequently, any open ball of radius r has at most λ vertices and $G \in \mathfrak{G}^{r,\lambda}$.

For every lattice point $p \in (1-\epsilon)\cdot\mathbb{Z}^2$, the vertices in $\mathcal{B}_{\epsilon/2}(p)$ form a clique. We know that, as a geometric graph, $(1-\epsilon)\cdot\mathbb{Z}^2$ is isomorphic to $\mathbb{Z}^2$. As a result, if p and q are two lattice points that are adjacent in $(1-\epsilon)\cdot\mathbb{Z}^2$, then any vertex in $\mathcal{B}_{\frac{\epsilon}{2}}(p)$ is adjacent to any vertex in $\mathcal{B}_{\frac{\epsilon}{2}}(q)$. Assuming that ϵ is sufficiently small, these are the only adjacencies that we have in G. Considering all the vertices that we distributed around each lattice point as a mega-vertex and the edges that join the vertices in one mega-vertex to another mega-vertex as a mega-edge, we will have a graph isomorphic to $\mathbb{Z}^2$. In order to protect a mega-vertex from catching and spreading the fire, we need to protect all the λ vertices it contains. Since two firefighters is the necessary condition to stop any finite fire in the square lattice [3], we have $f^{r,\lambda} \geq 2\lambda$. It is not hard to check that in this interval $2\lambda \geq 2\lfloor\frac{\lambda}{2r^2}\rfloor$.

When $\frac{\sqrt{2}}{9} \leq r < \frac{\sqrt{2}}{2}$, the proof of the inequality is similar to that of the previous case. The only differences are: $0 < \epsilon < \frac{\sqrt{2}}{2} - 2r$; instead of $\mathbb{Z}^2_\epsilon$, we use the square lattice with side length $\frac{\sqrt{2}}{2} - \epsilon$. As a geometric graph, this lattice is isomorphic to the strong square lattice $\mathbb{Z}^2_\boxtimes$. We know from [6] that 4 firefighters is the necessary condition to stop any finite fire in the strong square lattice; therefore, $f^{r,\lambda} \geq 4\lambda$. It is easy to check that $4\lambda \geq 4\lfloor\frac{2\lambda}{81r^2}\rfloor$ in this interval.

Finally, let $0 < r < \frac{\sqrt{2}}{9}$. Suppose $\frac{\sqrt{2}}{4}\cdot\mathbb{Z}^2$ be the square lattice with sides of length $\frac{\sqrt{2}}{4}$. For $(i,j) \in \mathbb{Z}^2$, define

$$\mathcal{S}_{i,j} = \{(x,y) \mid \frac{\sqrt{2}}{4}i \leq x \leq \frac{\sqrt{2}}{4}(i+1), \frac{\sqrt{2}}{4}j \leq y \leq \frac{\sqrt{2}}{4}(j+1)\},$$

which are the square faces of $\frac{\sqrt{2}}{4}\cdot\mathbb{Z}^2$. Let $\mathfrak{S}$ be the set of all these squares. Inside each square $\mathcal{S}_{i,j}$, consider the square $\mathcal{T}_{i,j}$ whose distance from the boundary of $\mathcal{S}_{i,j}$ on each side is r, i.e., $\mathcal{T}_{i,j}$ is a square centered inside $\mathcal{S}_{i,j}$ of side length $\frac{\sqrt{2}}{4} - 2r$. Let $\frac{\sqrt{2}}{4} - 2r > \epsilon > 0$ be a very small real and let n be the largest positive integer such that $(n-1)2r + n\epsilon \leq \frac{\sqrt{2}}{4} - 2r$, i.e., $n = \lfloor\frac{\sqrt{2}}{4(2r+\epsilon)}\rfloor$. This means that we can construct a $n \times n$ grid of open balls of radius ϵ inside $\mathcal{T}_{i,j}$ such that distance between any two of these open balls is at least $2r$. Denote the set of all these open balls inside $\mathcal{T}_{i,j}$ by $\mathfrak{B}_{i,j}$ and we will randomly choose λ distinct points in each of them. Let G be the geometric graph whose vertices are these randomly chosen points in each open ball in $\mathfrak{B}_{i,j}$, for all $i,j \in \mathbb{Z}$. We will show that $G \in \mathfrak{G}^{r,\lambda}$. Since the distance of $\mathcal{T}_{i,j}$ from the boundary of $\mathcal{S}_{i,j}$ from either side is r, the distance between any open ball in $\mathfrak{B}_{i,j}$ and an open ball in any other $\mathfrak{B}_{i',j'}$ is at least $2r$. Moreover, the distance between any two open ball in $\mathfrak{B}_{i,j}$ is at least $2r$, and as a result, around each open ball in $\mathfrak{B}^{r,\lambda}$ there is an annulus of annular width $2r$ that is empty of any vertices in G. It follows that any open ball of radius r can only contain vertices that belong to exactly one of the open balls in $\mathfrak{B}_{i,j}$. Consequently, any open ball of radius r has at most λ vertices and $G \in \mathfrak{G}^{r,\lambda}$.

Since the side of each square is $\frac{\sqrt{2}}{4}$, every vertex in $\mathcal{S}_{i,j}$ is adjacent to every vertex in $\mathcal{S}_{l,k}$, where $l \in \{i-1, i, i+1\}$ and $k \in \{j-1, j, j+1\}$, i.e., any vertex in $\mathcal{S}_{i,j}$ is adjacent to any other vertex in $\mathcal{S}_{i,j}$ and the eight squares touching it. Note that these are not all the possible adjacencies in G. Based on our construction of G, we know that each $\mathcal{S}_{i,j}$ contains $n^2\lambda$ vertices. Considering all the vertices in $\mathcal{S}_{i,j}$ as a mega-vertex and the edges joining these vertices to the vertices in $\mathcal{S}_{i,j+1}$ (also, to $\mathcal{S}_{i,j-1}$, $\mathcal{S}_{i-1,j}$, and $\mathcal{S}_{i+1,j}$) as a mega-edge, we have a mega-graph that is isomorphic to the strong square lattice $\mathbb{Z}^2_\boxtimes$. Again, we must protect all the vertices in the mega-vertex. Since four firefighters are necessary to stop any finite fire in the strong square lattice [6], we have $f^{r,\lambda} \geq 4n^2\lambda$. Since $\frac{\sqrt{2}}{4(2r+\epsilon)} > \frac{\sqrt{2}}{9r}$, we have $f^{r,\lambda} \geq 4\lfloor\frac{\sqrt{2}}{9r}\rfloor^2\lambda$. It is not hard to

prove that when $x \geq 1$, $\lfloor x \rfloor \geq \frac{x}{2}$. Since $\sqrt{2}/(9r) > 1$ in this interval, we have

$$4\Big\lfloor \frac{\sqrt{2}}{9r} \Big\rfloor^2 \lambda \geq \frac{2}{81r^2}\lambda \geq \Big\lfloor \frac{2\lambda}{81r^2} \Big\rfloor. \qquad \square$$

3.2 Lower Bounded Density

Before we proceed to prove Theorem 3.6, we will prove the following lemma.

Lemma 3.5. *For all n, let $\mathcal{E}_n$ be the circle centered at $\mathbf{0}$ and of radius $n(1+\epsilon)$ where $\frac{1}{2} \geq \epsilon > 0$. We divide every $\mathcal{E}_n$ into*

$$s_n = \Big\lceil \frac{2\pi(\kappa+1)n}{\epsilon} \Big\rceil$$

equal arcs, as shown in Figure 2. If $\mathfrak{P}$ denotes the set of endpoints of these arcs for all n, then any open ball of radius $(\frac{1}{2}+\epsilon)$ must contain at least κ elements in $\mathfrak{P}$.

Proof. (of Lemma 3.5) Let $\mathcal{B}$ be an open ball of radius $(\frac{1}{2}+\epsilon)$ and center c. We know that c is either on one of the $\mathcal{E}_n$'s or is between two consecutive circles. Without loss of generality, we assume that c is equidistant from $\mathcal{E}_m$ and $\mathcal{E}_{m+1}$ for some m. If not, it would be closer to one of these circles, and, as a result, it will separate a larger sector from the closer circle. Suppose $\mathcal{L}$ and $\mathcal{L}''$ are line segments connecting the intersection points of $\mathcal{B}$ with $\mathcal{E}_{m+1}$ and $\mathcal{E}_m$, respectively. Also, let $\mathcal{L}'$ be the line segment tangent to $\mathcal{E}_m$ parallel to $\mathcal{L}$ and $\mathcal{L}''$ (see Figure 3).

Let x be the distance between c and $\mathcal{L}$, and let y be half of the length of $\mathcal{L}$. Then $x^2+y^2 = (\frac{1}{2}+\epsilon)^2$. On the other hand, we have $x < \frac{1+\epsilon}{2}$. These two facts imply that $y^2 > \frac{\epsilon}{2}(1+\frac{3}{2}\epsilon) > \frac{\epsilon}{2}(1+\epsilon)$. But $y < \frac{1}{2}+\epsilon \leq 1$ which enables us to write $y > y^2 > \frac{\epsilon}{2}(1+\epsilon)$. Let $\mathcal{K}$ be the sector of $\mathcal{E}_{m+1}$ inside B and k its the arc length. Since a line segment between two points has the shortest arc length among all possible curves connecting the two points, then $\frac{k}{2} \geq y$. As a result, $\frac{k}{2} > \frac{\epsilon}{2}(1+\epsilon)$ which implies

$$\frac{2\pi(m+1)(1+\epsilon)}{k} < \frac{2\pi(m+1)}{\epsilon}$$

or, equivalently,

$$\frac{2\pi(m+1)(1+\epsilon)}{\frac{k}{\kappa+1}} < \frac{2\pi(\kappa+1)(m+1)}{\epsilon}.$$

The denominator of the left-hand side of this inequality is the arc length of the smaller sectors we get from dividing $\mathcal{K}$ into $\kappa+1$ equal arcs. This guarantees that $\mathcal{B}$ contains at least κ endpoints of these arcs. Dividing the perimeter of $\mathcal{E}_{m+1}$ into sectors of arc length $\frac{k}{\kappa+1}$, gives us a lower bound for the number of equal arcs we need to divide $\mathcal{E}_{m+1}$ so that at least κ of these endpoints is in $\mathcal{B}$. Since

$$\Big\lceil \frac{2\pi(m+1)(1+\epsilon)}{\frac{l}{\kappa+1}} \Big\rceil \leq \Big\lceil \frac{2\pi(\kappa+1)(m+1)}{\epsilon} \Big\rceil,$$

if we divide $\mathcal{E}_{m+1}$ into s_{m+1} equal arcs, $\mathcal{B}$ is guaranteed to contain at least κ of the endpoints of these arcs.

Similarly, define x', y' and x'', y'' for $\mathcal{L}'$ and $\mathcal{L}''$, respectively. We will first show that $x' > \frac{1}{2}$. Suppose the converse is true. This implies that $1+\epsilon-\frac{1}{2} \leq 1+\epsilon-x'$, and, as a result, we have $\frac{1}{2}+\epsilon+x' \leq 1+\epsilon$. Since $(1+\epsilon)$ is the distance between $\mathcal{E}_m$ and $\mathcal{E}_{m+1}$, this inequality

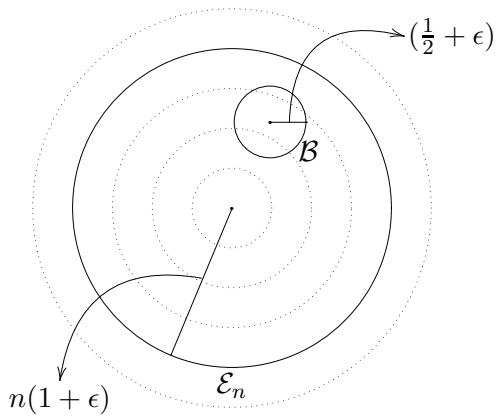

Figure 2: A Circle of Radius $(\frac{1}{2}+\epsilon)$ Containing at Least κ Points in $\mathfrak{P}$

implies that $\mathcal{E}_{m+1}$ and $\partial\mathcal{B}$ can intersect in at most one point. This is in contradiction since c is equidistant from both circles and the diameter of $\mathcal{B}$ is slightly bigger $1+\epsilon$ which forces $\partial\mathcal{B}$ to intersect $\mathcal{E}_m$ and $\mathcal{E}_{m+1}$ in two points each.

For some t, s, $x'+t=\frac{1}{2}+\epsilon$ and $x''=x'+s$. It follows that $t<\epsilon$ and $s\leq\frac{t}{2}$. Consequently, $x''<\frac{1}{2}+2\epsilon$. Since $(x'')^2+(y'')^2=(\frac{1}{2}+\epsilon)^2$, we have

$$\frac{k'}{2}>y''>(y'')^2>\epsilon(1+3\epsilon)>\frac{\epsilon}{2}(1+\epsilon),$$

where k' is the arc length of the sector of $\mathcal{E}_m$ inside $\mathcal{B}$. As a result, we have

$$\frac{2\pi m(1+\epsilon)}{k'}<\frac{2\pi m}{\epsilon}$$

or, equivalently,

$$\frac{2\pi m(1+\epsilon)}{\frac{k'}{\kappa+1}}<\frac{2\pi(\kappa+1)m}{\epsilon}.$$

Similar to the argument we gave for $\mathcal{E}_{m+1}$, dividing $\mathcal{E}_m$ into s_m equal arcs guarantees that $\mathcal{B}$ contains κ endpoints of these arcs on $\mathcal{E}_m$.

Finally, if we divide each $\mathcal{E}_n$ into s_n equal arcs, any open ball of radius $\frac{1}{2}+\epsilon$ must contain at least κ of the endpoints of these arcs. □

To prove Theorem 2.6, we will prove the following proposition. One can easily check that for $C=\frac{\sqrt{2}}{8}$, the statement of Theorem 2.6 gives smaller lower bounds than the ones given for each particular interval in Proposition 3.6.

Proposition 3.6. *Assuming that $\kappa\geq 1$,*

1. $f_{r,\kappa}=0$, *for* $\frac{1}{2}<r$;
2. $f_{r,\kappa}\geq 1$, *for* $\frac{1}{4}<r\leq\frac{1}{2}$;
3. $f_{r,\kappa}\geq 3\kappa$, *for* $\frac{\sqrt{5}}{10}\leq r\leq\frac{1}{4}$;
4. $f_{r,\kappa}\geq 2\lfloor\frac{\sqrt{5}}{10r}\rfloor^2\kappa$, *for* $\frac{\sqrt{2}}{8}\leq r<\frac{\sqrt{5}}{10}$;

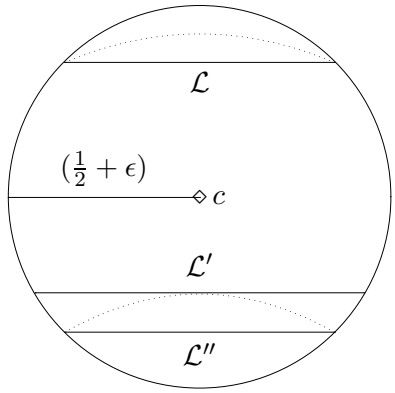

Figure 3: $\mathcal{B}$, $\mathcal{E}_m$, and $\mathcal{E}_{m+1}$

5. *$f_{r,\kappa} \geq 4\lfloor \frac{\sqrt{2}}{8r} \rfloor^2 \kappa$, for $0 < r < \frac{\sqrt{2}}{8}$.*

Proof. For $\frac{1}{2} < r \leq 1$, we can use the following graph: let $\mathcal{E}_n = \mathcal{C}_{r_n}$ where $r_n = n(1+\epsilon)$ and $\frac{1}{2} \geq \epsilon > 0$. We divide every $\mathcal{E}_n$ into $\lceil \frac{2\pi(\kappa+1)n}{\epsilon} \rceil$ equal sectors, and suppose the vertices of the graph G are the origin and the endpoints of each sector marked on each $\mathcal{E}_n$. By Lemma 3.5, we know that every open ball of radius $\frac{1}{2} < r \leq 1$ contains at least κ vertices of G; therefore, $G \in \mathfrak{G}_{r,\kappa}$. Moreover, the origin is not adjacent to any of the vertices in $\mathcal{E}_1$ since their distance is more than one. Similarly, no vertices in $\mathcal{E}_m$ are adjacent to a vertex in $\mathcal{E}_{m+1}$. Consequently, the vertices that belong to a component of G are on the same circle. Since there are finitely many vertices on each circle, each component is finite. As a result, no firefighters are needed to stop a finite fire in G and we have $f(G) = 0$. This proves that $f_{r,\kappa} = 0$. Note that this construction also works in the case when $r > 1$ since every open ball of radius r contains an open ball of radius one.

When $\frac{1}{4} < r \leq \frac{1}{2}$, we will demonstrate that each vertex of $G \in \mathfrak{G}^{r,\kappa}$ belongs to an infinite component. Let v_0 be an arbitrary vertex in G, and let $\mathcal{L}$ be a fixed line that goes through v_0. Let $\mathcal{F}_0$ be an open ball of radius r whose center is on $\mathcal{L}$ and has v_0 on its boundary. Since $G \in \mathfrak{G}_{r,\kappa}$, $\mathcal{F}_0$ must contain at least a vertex v_1 different than v_0. Since the diameter of open ball circle is $2r$, the distance between v_0 and v_1 is less than one, and, as a result, they are adjacent vertices. Let $\mathcal{L}_1$ be the perpendicular line to $\mathcal{L}$ that goes through v_1. In the half plane that does not contain v_0, consider the open ball $\mathcal{F}_1$ of radius r which is tangent to $\mathcal{L}_1$ at v_1 (see Figure 4). Recursively, we find an infinite path in G: For all $i \geq 1$, we assume that we have a vertex v_i and a line $\mathcal{L}_i$ perpendicular to $\mathcal{L}$ that goes through v_i. In the half plane that does not contain v_j, for $1 \leq j \leq i$, consider the open ball $\mathcal{F}_i$ which is tangent to $\mathcal{L}_i$ at v_i. It follows that $\mathcal{F}_i$ must contain a vertex v_{i+1} which will be adjacent to v_i. This construction shows that G contains an infinite path, and, as a result, $f(G) \geq 1$. Since this is true for any $G \in \mathfrak{G}_{r,\kappa}$, we have $f_{r,\kappa} \geq 1$.

Now let $\frac{\sqrt{5}}{10} \leq r \leq \frac{1}{4}$. Let $\mathbb{T}_t$ be the triangular lattice whose faces are equilateral triangles with side t. Consider $\mathbb{T}_{\frac{1}{2}}$. We will pack the plane using open balls of radius $\frac{1}{4}$ whose centers are the vertices of faces of $\mathbb{T}_{\frac{1}{2}}$. Let us denote the set of all these open balls by $\mathfrak{B}$. On one hand, every open ball in $\mathfrak{B}$ contains an open ball of radius r, and, as a result, it must contain at least κ vertices. On the other hand, all the vertices in $\mathcal{B}_1 \in \mathfrak{B}$ are adjacent to the vertices in $\mathcal{B}_2 \in \mathfrak{B}$, provided that the centers of $\mathcal{B}_1$ and $\mathcal{B}_2$ are the two endpoints of an edge in $\mathbb{T}_{\frac{1}{2}}$. Considering all the vertices in an open ball in $\mathfrak{B}$ as a mega-vertex and the edges joining these vertices to the vertices of a neighboring open ball as a mega-edge, we have a mega-graph that

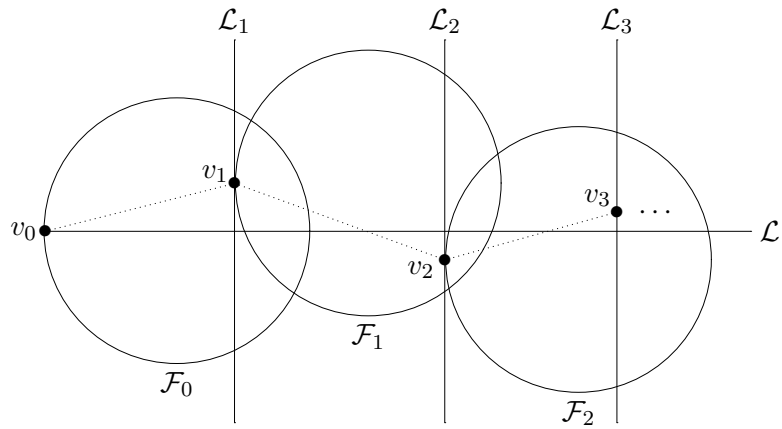

Figure 4: Finding an Infinite Path in G

is isomorphic to the triangular lattice. We know from [3, 6] that three firefighters are necessary and sufficient to contain any finite fire in the triangular lattice. In order to protect a mega-vertex from catching and spreading fire, we need to protect all the vertices it contains. Since each mega-vertex contains at least κ vertices, we need at least 3κ firefighters to stop any finite fire in G. It follows that $f_{r,\kappa} \geq 3\kappa$.

Now let $\frac{\sqrt{2}}{8} \leq r < \frac{\sqrt{5}}{10}$. For $(i,j) \in \mathbb{Z}^2$, define

$$\mathcal{S}_{i,j} = \{(x,y) \mid \frac{\sqrt{5}}{5}i \leq x < \frac{\sqrt{5}}{5}(i+1), \frac{\sqrt{5}}{5}j \leq y < \frac{\sqrt{5}}{5}(j+1)\}.$$

Since the side of each square is $\frac{\sqrt{5}}{5}$, every vertex in $\mathcal{S}_{i,j}$ is adjacent to every vertex in $\mathcal{S}_{i,j+1}$, $\mathcal{S}_{i,j-1}$, $\mathcal{S}_{i+1,j}$, and $\mathcal{S}_{i-1,j}$. Let $\mathfrak{S}$ be the set of all these squares. Now each square in $\mathfrak{S}$ contains at least $\left\lfloor \frac{\sqrt{5}}{10r} \right\rfloor^2$ squares of side length $2r$. We know that each square of side length $2r$ circumscribes a circle of radius r, and, as a result, it contains at least κ vertices. It follows that every square in $\mathfrak{S}$ contains at least $\left\lfloor \frac{\sqrt{5}}{10r} \right\rfloor^2 \kappa$ vertices. Considering all the vertices in $\mathcal{S}_{i,j}$ as a mega-vertex and the edges joining these vertices to the vertices in $\mathcal{S}_{i,j+1}$ (also, to $\mathcal{S}_{i,j-1}$, $\mathcal{S}_{i-1,j}$, and $\mathcal{S}_{i+1,j}$) as a mega-edge, we have a mega-graph that is isomorphic to the square lattice. In order to protect a mega-vertex from catching and spreading fire, we need to protect all the vertices it contains. Since two firefighters is the necessary and sufficient condition to stop any finite fire in the square lattice [3], we have $f_{r,\kappa} \geq 2\left\lfloor \frac{\sqrt{5}}{10r} \right\rfloor^2 \kappa$.

Finally, let $0 < r < \frac{\sqrt{2}}{8}$. For $(i,j) \in \mathbb{Z}^2$, define

$$\mathcal{T}_{i,j} = \{(x,y) \mid \frac{\sqrt{2}}{4}i \leq x < \frac{\sqrt{2}}{4}(i+1), \frac{\sqrt{2}}{4}j \leq y < \frac{\sqrt{2}}{4}(j+1)\}.$$

Since the side of each square is $\frac{\sqrt{2}}{4}$, every vertex in $\mathcal{T}_{i,j}$ is adjacent to every vertex in $\mathcal{T}_{l,k}$, where $l \in \{i-1, i, i+1\}$ and $k \in \{j-1, j, j+1\}$, i.e., any vertex in $\mathcal{T}_{i,j}$ is adjacent to any other vertex in $\mathcal{T}_{i,j}$ and the eight squares touching it. Let $\mathfrak{T}$ be the set of all these squares. We know that each square in $\mathfrak{T}$ contains at least $\left\lfloor \frac{\sqrt{2}}{8r} \right\rfloor^2$ squares of side length $2r$. Also, each square of side length $2r$ circumscribes a circle of radius r, and, as a result, it contains at least κ vertices. It follows that every square in $\mathfrak{T}$ contains at least $\left\lfloor \frac{\sqrt{2}}{8r} \right\rfloor^2 \kappa$ vertices. Considering all the vertices in $\mathcal{T}_{i,j}$ as a mega-vertex and the edges joining these vertices to the vertices in

the eight squares touching it as a mega-edge, we have a mega-graph that is isomorphic to the strong square lattice $\mathbb{Z}^2_{\boxtimes}$. In order to protect a mega-vertex from catching and spreading fire, we need to protect all the vertices it contains. Since 4 firefighters is the necessary and sufficient condition to stop any finite fire in the strong square lattice [6], we have $f_{r,\kappa} \geq 4\lfloor \frac{\sqrt{2}}{8r} \rfloor^2 \kappa$. $\square$

We observed in the first statement of Proposition 3.6 that if $r > 1/2$ then $f_{r,\kappa} = 0$, that is, we *might* not need any firefighters. But a critic might complain that the graphs constructed in the proof have arbitrarily large components, so that there is no bound on how far an unchecked fire might spread. We conclude this work by computing the value of r at which this ceases to be an issue.

Theorem 3.7. *For $r < \frac{\sqrt{3}}{3}$, $G \in \mathfrak{G}_{r,\kappa}$ has arbitrarily large components. On the other hand, when $r \geq \frac{\sqrt{3}}{3}$, $G \in \mathfrak{G}_{r,\kappa}$ might have components whose size is uniformly bounded.*

We prove the first statement of this theorem by contradiction. Suppose $\{C_i\}_{i\in\mathbb{N}}$ in the set of all components of G and we assume that the number of vertices in each component is bounded by some fixed number $M > 0$. First, we will justify the fact the set of components of G is countably infinite: If C_i is a component in G and x_i a vertex in C_i, $\mathcal{B}_1(x_i)$ can only contain vertices from C_i. To each component C_i of G, we assign a unit open ball centered at a vertex x_i in C_i. Based on the above observation, none of these balls intersect at a vertex. If we assume that there are uncountably many components in G, there will be uncountably many open balls. Since there are uncountably many of these centers, a subset $\{y_j\}_{j\in\mathbb{N}}$ of them must have an accumulation point y_0. Then for some $N > 0$, $|y_0 - y_j| < \frac{1}{2}$ for all $j > N$. Since they are all in the same component of G, we have a contradiction. Since we can put infinitely many non-intersecting open balls of radius r with centers along a line, the vertex set of G has to be infinite. Since each component has finitely many vertices, the set of components of G is countably infinite.

Now define $\mathcal{V}_i = \{p \in \mathbb{R}^2 \mid \rho(p, C_i) < \rho(p, C_j), \forall j \neq i\}$ to be the open Voronoi cell associated to C_i, and let $\mathcal{V} = \mathbb{R}^2 \backslash (\bigcup_{i\in\mathbb{N}} \mathcal{V}_i)$ be the Voronoi diagram associated to the components of G. Similarly, define $\mathcal{V}_x = \{p \mid \rho(p, x) < \rho(p, y), \forall y \in G\backslash\{x\}\}$ to be the open Voronoi cell associated to a vertex x and let $\mathcal{V}_G = \mathbb{R}^2 \backslash (\bigcup_{x\in G} \mathcal{V}_x)$ be the Voronoi diagram associated to the vertices of G. Given these definitions and the assumption that the cardinality of each component is uniformly bounded, the following facts are not hard to prove:

1. For all i, the Euclidean diameter of $\mathcal{V}_i$ is uniformly bounded.
2. For all i, $\mathrm{Cl}(\mathcal{V}_i) = \{p \in \mathbb{R}^2 \mid \rho(p, C_i) \leq \rho(p, C_j), \forall j \neq i\}$.
3. For all i, $\mathcal{V}_i$ and $\mathrm{Cl}(\mathcal{V}_i)$ are path-connected, and, as a result, connected.
4. For all i, $\mathcal{V}_i$ is open.
5. For all i, $\partial\,\mathcal{V}_i = \{p \in \mathrm{Cl}(\mathcal{V}_i) \mid \exists j \neq i, \rho(p, C_i) = \rho(p, C_j)\}$ and $\mathcal{V} = \bigcup_{i\in\mathbb{N}} \partial\,\mathcal{V}_i$.
6. For all i, there exists some $j \neq i$ such that $\partial\,\mathcal{V}_i \cap \partial\,\mathcal{V}_j \neq \varnothing$.
7. Any bounded region $\mathcal{R}$ in the plane contains finitely many vertices in G.
8. For all $x \in G$, $\mathrm{Cl}(\mathcal{V}_x)$ is a convex polygon.
9. For all i, $\partial\,\mathcal{V}_i$ has finitely many connected topological components.

10. For all i, the complement of $\mathrm{Cl}(\mathcal{V}_i)$ has a unique connected topological component $\mathcal{E}_i$ that contains infinitely many vertices.

Before we proceed to prove Theorem 3.7, we need to prove the following lemma which we will use in proving the first statement of this theorem:

Lemma 3.8. *There exist distinct i, j, k such that $\partial \mathcal{V}_i \cap \partial \mathcal{V}_j \cap \partial \mathcal{V}_k$ is not empty.*

Proof. Suppose not. Since $\mathrm{Cl}(\mathcal{V}_x)$ is a convex polygon, $\partial \mathcal{V}_x$ is a finite union of line segments which we call the Voronoi edges. It can be shown that these Voronoi edges are line segments on the perpendicular bisector of exactly two vertices of G. If we call the nonempty intersection of any two of these edges a Voronoi vertex, then the Voronoi vertices are where the perpendicular bisector of three or more vertices intersect; hence, the Voronoi vertices in $\mathcal{V}_G$ are equidistant from at least three vertices in G. Moreover, every Voronoi edge contains exactly two Voronoi vertices, namely its endpoints (see Proposition 5.3.2 in [5] on p. 87).

For all $x \in G$ and i, if $x \in C_i$, then label $\mathcal{V}_x$, i. Let e be a Voronoi edge in $\partial \mathcal{E}_i \subseteq \partial \mathrm{Cl}(V_i)^c = \partial \mathrm{Cl}(V_i) = \partial \mathcal{V}_i$. It follows that there exist $j \neq i$ and $y \in C_j$ such that e is a line segment on the perpendicular bisector of y and some $x \in C_i$. Consequently, e has the label i on one side and j on the other. Let v and u be the two endpoints of e. Suppose v is equidistant from x, y, and at least one other vertex z. Then z has to be either in C_i or C_j; otherwise, $v \in \partial \mathcal{V}_i \cap \partial \mathcal{V}_j \cap \partial \mathcal{V}_k$ for some k distinct from i and j. Consider the closed disk $\mathcal{D}_{\rho(v,x)}(v)$ which has all the vertices that are closest to v on its boundary. Starting at x and writing the labels of the cells which have v on their boundary in a counterclockwise order, we have a finite sequence of the form $(t_1, t_2, \ldots, t_m)$, where $t_l \in \{i, j\}$. We will show that $(t_1, t_2, \ldots, t_m) = (i, i, \ldots, i, j, j, \ldots, j)$. Suppose not and the subsequence i, j, i, j must occur. Let the vertices x_1, y_1, x_2, y_2 be the vertices in $\mathcal{C}_{\rho(v,x)}(v)$ associated to this subsequence. Since $\rho(x_1, y_1)$ and $\rho(y_1, x_2)$ are greater than one, by looking at the sectors they mark on $\mathcal{C}_{\rho(v,x)}(v)$, we know $\rho(x_1, x_2) > 1$. Similarly, $\rho(y_1, y_2) > 1$. Both C_i and C_j are connected graphs, but since x_1 and x_2 prevent y_1 and y_2 to be connected to each other via the vertices in $C_j \cap \mathcal{C}_{\rho(v,x)}(v)$, y_1 and y_2 are connected via a path not in $\mathcal{D}_{\rho(v,x)}(v)$. The same is true about x_1 and x_2. But this forces the two paths to intersect at a point, which is a contradiction since $\mathcal{V}_i \cap \mathcal{V}_j = \varnothing$.

Let us assume that the pair i, j in the middle of this cycle is the one associated to e, with x on one side and y on the other. But there will be another edge e_1 associated to the pair j, i at the end of this sequence, i.e., e_1 has an i on one side and a j on the other. In other words, $e_1 \in \partial \mathcal{V}_i \cap \partial \mathcal{V}_j$ and has v as an endpoint. Let v_1 be the other endpoint of e_1. We continue in this fashion and will have a sequence $v_0 = v, e_1, v_2, e_2, v_3, \ldots$ of vertices and edges. Along this sequence, we do not return to any vertex; if we do, we would have the forbidden subsequences i, j, i, j or j, i, j, i at that vertex. We know that $\partial \mathcal{V}_i \subseteq \bigcup_{x \in C_i} \partial \mathcal{V}_x$, each $\mathrm{Cl}(\mathcal{V}_x)$ is a polygon, and each C_i has finite cardinality. Consequently, $\partial \mathcal{V}_i$ is made of finitely many Voronoi edges and, as a result, the aforementioned sequence has to end. The only possibility is u, the other endpoint of e, which implies that this sequence is a cycle $\mathcal{C}$ with i's on one side and j's on the other.

By Jordan curve theorem, $\mathcal{C}$ separates the plane into two open and connected regions, an interior region $\mathcal{E}$ bounded by $\mathcal{C}$ and an exterior region $\mathcal{E}$, and any continuous path connecting a point of $\mathcal{E}$ to a point of $\mathcal{E}$ intersects with $\mathcal{C}$ at some point along the cycle. Since any bounded region can only contain finitely many vertices, $\mathcal{E}_i$ is in $\mathcal{E}$. Moreover, $\mathcal{V}_i$ and $\mathcal{V}_j$ are, respectively, in the interior and the exterior region since they are open and path-connected. Also, $\mathrm{Cl}(\mathcal{V}_i)$ does not intersect with the exterior region: if it does, this happens at some point $z \in \partial V_i$. We know every open neighborhood of z intersect with $\mathcal{V}_i$. On the other hand, since $\mathcal{E}$ is open, there

is a neighborhood of z that is entirely in $\mathcal{E}$. This is not possible because $\mathcal{V}_i \cap \mathcal{E} \subseteq \mathcal{I} \cap \mathcal{E} = \emptyset$. It follows that $\mathrm{Cl}(\mathcal{V}_i) \subseteq \mathcal{C} \cup \mathcal{I}$ and consequently, $\mathcal{E} \subseteq \mathrm{Cl}(\mathcal{V}_i)^c$. Since $\mathcal{I}$ is bounded, it contains finitely many vertices in G, and as a result, $\mathcal{E}$ must contains infinitely many. It follows that $\mathcal{E}$ is equal to $\mathcal{E}_i$ because $\mathcal{E}$ is connected and $\mathcal{E}_i$ is the unique connected component of $\mathrm{Cl}(\mathcal{V}_i)^c$ containing infinitely many vertices. Consequently, $\mathcal{E}_i^c$ is compact (a closed and bounded subset of $\mathbb{R}^2$) and $\mathcal{V}_j \subset \mathcal{E}_i$. Finally, $\partial\,\mathcal{E}_i = \partial\,\mathcal{V}_i \cap \partial\,\mathcal{V}_j$ since we have i's on one side of $\mathcal{C}$ and j's on the other and $\partial\,\mathcal{E}_i = \mathcal{C}$.

Similarly, $\mathcal{V}_j$ is in the interior of $\partial\,\mathcal{E}_j$ and for some k, $\mathcal{V}_k \subset \mathcal{E}_j$ is in its exterior, i.e., $\partial\,\mathcal{E}_j = \partial\,\mathcal{V}_j \cap \partial\,\mathcal{V}_k$. Based on our assumption, $\partial\,\mathcal{E}_i \cap \partial\,\mathcal{E}_j = \varnothing$. Since the open set $\mathcal{V}_j$ is in the exterior of $\partial\,\mathcal{E}_i$ and in the interior of $\partial\,\mathcal{E}_j$, $\partial\,\mathcal{E}_j$ is in the exterior of $\partial\,\mathcal{E}_i$, i.e., $\mathcal{E}_j \subset \mathcal{E}_i$ and $\mathcal{E}_i^c \subset \mathcal{E}_j^c$. Since $\mathcal{E}_i^c$ and $\mathcal{E}_j^c$ are compact, $\mathrm{diam}(\mathcal{E}_i^c) = \rho(p,q)$ for some $p,q \in \partial\,\mathcal{E}_i$ and $\mathrm{diam}(\mathcal{E}_j^c) = \rho(p',q')$ for some $p',q' \in \partial\,\mathcal{E}_i$. Since $\partial\,\mathcal{E}_j$ is in the exterior of $\partial\,\mathcal{E}_i$ and they do not intersect, the line $\mathcal{L}$ going through p and q will intersect $\partial\,\mathcal{E}_j$ at two points, u and v, whose distance will be strictly greater than $\rho(p,q)$. It follows that $\mathrm{diam}(\mathcal{E}_i^c) = \rho(p,q) < \rho(u,v) \leq \rho(p',q') = \mathrm{diam}(\mathcal{E}_j^c)$. Consequently, there will be a sequence $i_0 = i, i_1 = j, i_2, \ldots$ such that $\mathcal{E}_{i_0}^c \subset \mathcal{E}_{i_1}^c \subset \mathcal{E}_{i_2}^c \subset \cdots$ and $\mathrm{diam}(\mathcal{E}_{i_0}^c) < \mathrm{diam}(\mathcal{E}_{i_1}^c) < \mathrm{diam}(\mathcal{E}_{i_2}^c) < \cdots$.

We will now show that for all l, $\mathrm{diam}(\mathcal{V}_l) = \mathrm{diam}(\mathcal{E}_l^c)$. Clearly, $\mathrm{diam}(\mathcal{V}_l) \leq \mathrm{diam}(\mathcal{E}_l^c)$. On the other hand, we know that $\mathrm{diam}(\mathcal{E}_l^c) = \rho(p,q)$ for some $p,q \in \partial\,\mathcal{E}_l$, but $\partial\,\mathcal{E}_l \subseteq \partial\,\mathcal{V}_l \subset \mathrm{Cl}(\mathcal{V}_l)$; therefore, $\rho(p,q) \leq \mathrm{diam}(\mathrm{Cl}(\mathcal{V}_l)) = \mathrm{diam}(\mathcal{V}_l)$. It follows that $\mathrm{diam}(\mathcal{V}_{i_0}) < \mathrm{diam}(\mathcal{V}_{i_1}) < \mathrm{diam}(\mathcal{V}_{i_2}) < \cdots$.

Finally, we will show that for every $\mathrm{diam}(\mathcal{V}_{i_l})$, there is some $k > l$ such that $\mathrm{diam}(\mathcal{V}_{i_l}) + 1 \leq \mathrm{diam}(\mathcal{V}_{i_k})$. If not, for all $k > l$, $\mathrm{diam}(\mathcal{V}_{i_k}) < \mathrm{diam}(\mathcal{V}_{i_l}) + 1$. This is not possible since any bounded region contains only finitely many vertices and $\bigcup_{k>l} \mathcal{V}_k$ contains infinitely many vertices. It follows that there is a subsequence $\{i_{l_m}\}$ of the sequence $\{i_l\}$ such that $\mathrm{diam}(\mathcal{V}_{i_{l_m}}) + 1 \leq \mathrm{diam}(\mathcal{V}_{i_{l_{m+1}}})$ which can not happen since the diameters of $\mathcal{V}_i$'s are uniformly bounded. □

Proof. (Theorem 3.7) Let us assume that there exists $M > 0$ such that the cardinality of each component is at most M. By using Lemma 3.8, we will demonstrate that there is an open ball of radius r that contains no vertices. We know that for some i,j,k, there exists $p \in \partial\,\mathcal{V}_i \cap \partial\,\mathcal{V}_j \cap \partial\,\mathcal{V}_k$. We will show that $\mathcal{B}_r(p)$ is empty of any vertices. Suppose not. Since $p \in \partial\,\mathcal{V}_i \cap \partial\,\mathcal{V}_j \cap \partial\,\mathcal{V}_k$, for some $s < r$, there exist x_i, x_j, and x_k in $\mathcal{V}_i$, $\mathcal{V}_j$, and $\mathcal{V}_k$, respectively, which belong to the circle of radius s and centered at p. Trisect this circle into equal sectors with x_i being the dividing point between two of the sectors. Then both x_j and x_k have to be in the sector opposite to x_i, otherwise their distance to x_i will be less than one. But this can not possible, because this forces the distance between x_j and x_k to be less than one. Since there exists an open ball of radius r which is empty of vertices, we have a contradiction. This proves the first claim of the theorem.

Now we will show that $r = \frac{\sqrt{3}}{3}$ is exact. We prove the claim for $\kappa = 1$. A similar construction works for $\kappa > 1$. Suppose $r > \frac{\sqrt{3}}{3}$. Recall that $\mathbb{T}_t$ is the triangular lattice whose faces are equilateral triangles with side t. Now consider a graph G whose vertices are the triangular vertices in $\mathbb{T}_1$. There are six vertices that are closest to a vertex v in G, but none of them are adjacent to v since the Euclidean distance between v and any of these vertex is equal to one. Consequently, each component of G is a singleton, and, as a result, their cardinality is uniformly bounded.

We will now demonstrate that every open ball $\mathcal{B}_r(p)$ contains at least a vertex in G. First, consider the case where p is on one of the triangular edges. Since the distance between the vertices on the two endpoints of this edge is one and $\mathcal{B}_r(p)$ has diameter $2r > 1$, $\mathcal{B}_r(p)$ must

contain at least one of these vertices. On the other hand, since the length of each triangular edge is one, we know that q, the intersection point of the perpendicular bisectors of the three vertices of a triangular face, is equidistant from them with this distance being $\frac{\sqrt{3}}{3}$. If $p = q$, then $\mathcal{B}_r(p)$ will contain at least three vertices. If not, p is closer to one of the vertices, and, yet again, $\mathcal{B}_r(p)$ will contain at least a vertex. $\square$

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