Taylor and Maclaurin Series

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Coefficients of a power series

Theorem: If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficient are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Taylor Series of a function about a

Suppose f is a function that can be represented by a power series, then

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \end{aligned}$$

Maclaurin Series

Maclaurin series are the Taylor series about $a = 0$

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 \frac{f'''(0)}{3!}x^3 + \cdots
\end{aligned}
$$

The Remainder and the Taylor polynomials T_n

The **Taylor polynomial**

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Let $R_n(x) = f(x) - T_n(x)$

Theorem: If $f(x) = T_n(x) + R_n(x)$ and

$$\lim_{n \longrightarrow \infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Taylor's Inequality

If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad |x-a| \leq d$$

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0 \qquad \text{for every real number } x.$$

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \qquad R = 1 \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R = \infty \\
\sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad R = \infty \\
\cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad R = \infty \\
\tan^{-1}(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad R = 1 \\
(1+x)^k = f(x) &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad R = 1
\end{aligned}
$$