Math 9 Fall 2003

Calculus of Functions of One and Several Variables, Honors

Instructors: V. Chernov, D. Little, A. Shumakovitch, M. Skandera

Homework Project I (20 points)

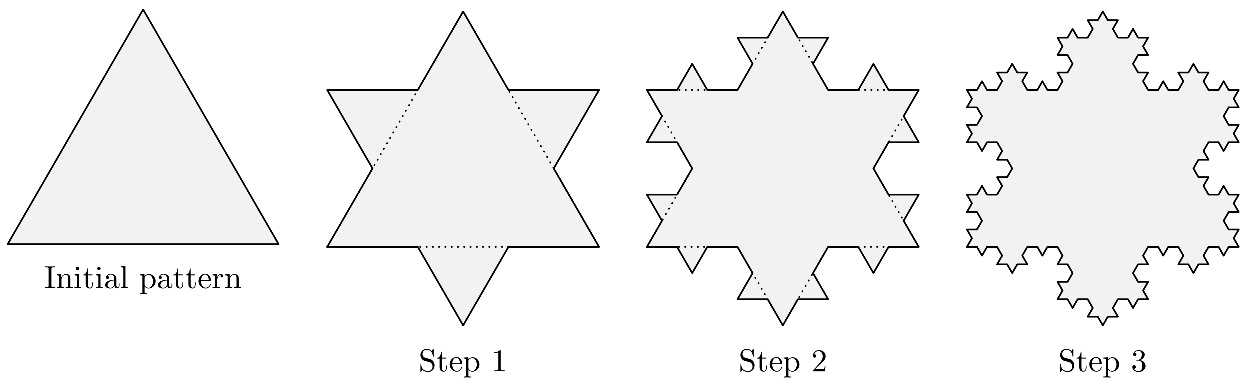
Fractal Curves and Their Properties

Due date: Wednesday, November 5, at the end of the lecture time

Description of the project

A fractal is a geometric shape that displays self-similarity on all scales. This means that you could zoom in on a part of it an infinite number of times and it would still look the same. Fractals can be easily found in nature. Clouds, mountains, river networks, and systems of blood vessels are all naturally occurring fractals. One of the most striking examples of fractal curves is a coastline. Imagine measuring its length with rulers of different length. It was observed long ago that the shorter the ruler, the longer the length measured. You will observe this paradox, known as **the coastline paradox**, as well while working on this project.

Mathematical fractals are often defined recursively. Consider an equilateral triangle with sides of length 1. The recursion step consists of taking each side of the current polygonal line, dividing it into three equal parts, and replacing the middle one with the two sides of a small equilateral triangle built on it. For example, after applying this procedure to the original equilateral triangle, one gets a David's star. The results of the first three steps are depicted below. The segments replaced are shown as dotted lines. Repeating the recursion step **infinite** number of times leads to a fractal curve, called the **Koch Snowflake**.



First three steps in construction of the Koch Snowflake

I. Let s_n, l_n, L_n, and A_n be, respectively, the number of sides, the length of a side, the total length, and the total area bounded by the n-th Koch curve, that is, the polygonal line obtained after the n-th iteration of the recursion step.

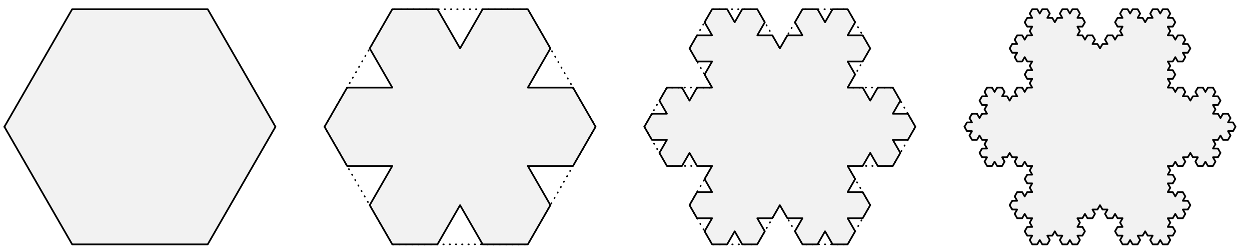
   a) Find formulas for s_n, l_n, L_n, and A_n.

   b) Show that $L_n \to \infty$ as $n \to \infty$.

   c) Show that A_n has a finite limit by computing the sum of an infinite series. Find this limit.

   d) Make a conclusion that the Koch Snowflake is infinitely long, yet encloses a finite area.

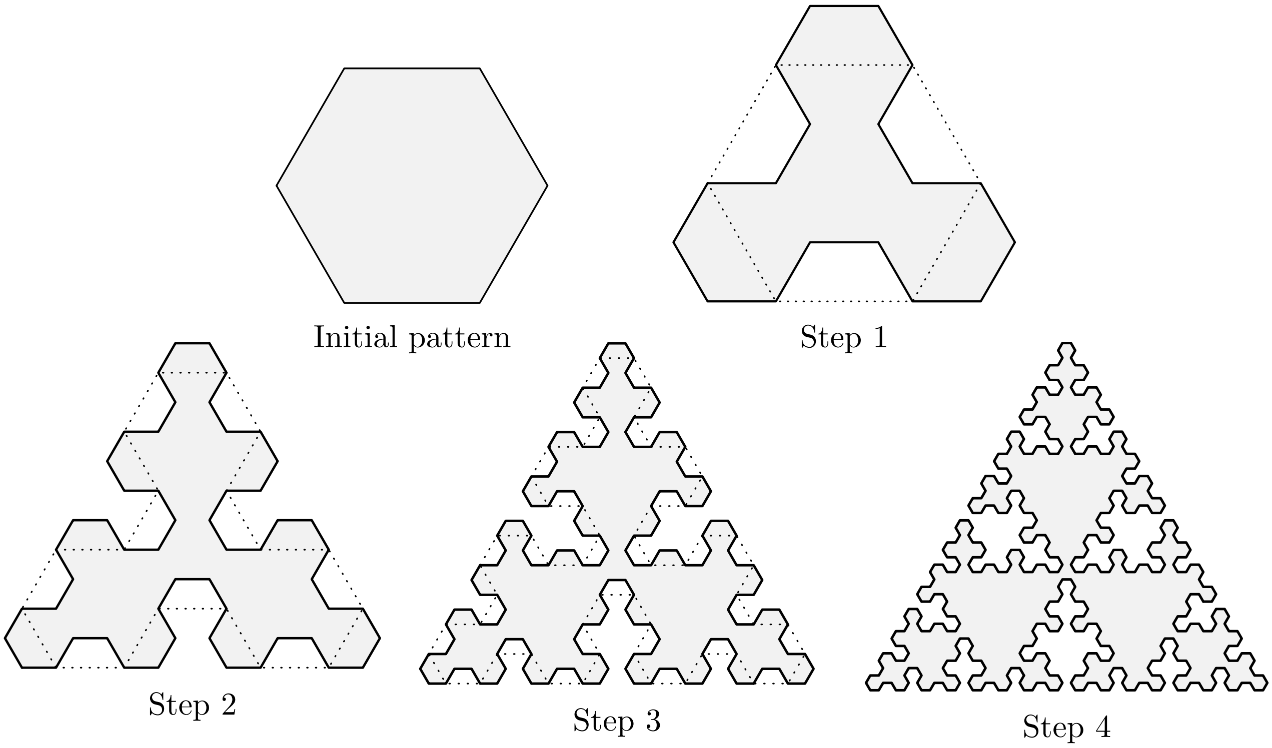
II. Investigate the Koch Exterior Snowflake (depicted below) similarly to Part I. Compare the results. In order to obtain the Koch Exterior Snowflake, one starts with a regular hexagon with sides of length $\frac{1}{\sqrt{3}}$ and applies the same procedure as in the case of the original Koch Snowflake, but with new sides always pointing inside and not outside of the snowflake.



Initial pattern Step 1 Step 2 Step 3

First three steps in construction of the Koch Exterior Snowflake

III. Investigate the Sierpinski Curve (depicted on the next page) similarly to Parts I and II. This curve is constructed starting with a regular hexagon with sides of length $\frac{1}{2}$. The recursion step consists of replacing each side of the polygonal line with three sides forming half a hexagon of an appropriate size. These sides are taken either inside or outside of the line, alternatively.



First four steps in construction of the Sierpinski Curve

Regulations concerning this project

1 Students can work on this project individually or in groups of **up to 4 persons.**

2 In case a group of students decides to work on the project together, **discussion of the problem and common solving inside of such a group is encouraged**. Members of different groups **should not share any essential information about the solution.**

3 In case a group of students decides to work together on the project, they are allowed to submit one solution for the group. This would imply that all the students in the group **do understand the presented solution** and can explain it to the professor in charge if asked to do so. In case you decide to submit one solution for all the members of the group please write clearly the names of all the group members on top of the solution. If one solution is submitted on behalf of a group of (up to 4) students, the grade for the project shall be **the same for all the members of the group.**

4 Students are allowed and encouraged **to consult mathematical books** and the Professors in charge.

5 The project is due by **Wednesday, November 5**, before the end of the lecture time. By that time the solution should be presented in **typed or clearly written form** to the Professor in charge. The solution should be presented **with full details**. Presenting only the answer is not sufficient.