Math 9 Fall 2002 – Group Projects

1. Consider the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + ...$, where the terms are the reciprocals of those natural numbers whose only prime divisors are 2, 3, and 5. Find the sum of the series.

2. Recall that the sine and cosine of angles like $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$ can be expressed explicitly and *exactly* in terms of radicals; thus, for example, $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$, while $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$. Find analogous formulas for $\cos(\frac{2\pi}{5})$ and $\sin(\frac{2\pi}{5})$ in terms of radicals. [Hint: Consider complex numbers.]

3. Define a sequence of integers a_n inductively by $a_0 = 0$, $a_n = 1$, and for $n > 1$, $a_n = a_{n-1} + a_{n-2}$; thus the sequence begins $0, 1, 1, 2, 3, 5, 8, 13, ...$ Prove that a_n is given by the remarkable formula

$$a_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n).$$

   (Note: It is not even apparent – and indeed it seems like a miracle – that the right-hand side is an integer!)

   [Hint: Consider the function f defined by the power series $f(x) = \Sigma_{n=0}^{\infty} a_n \frac{x^n}{n!}$. What happens when you differentiate f?]

4. (a) Determine all triples of positive integers (a, b, c) such that $c^2 = a^2 + b^2$. [Hint: This reduces to looking for all points on the unit circle whose coordinates are both rational. It may help to use a parametrization of the circle obtained by joining the point (x, y) on the circle to the point $(-1, 0)$ by a line segment; the y-intercept of this line segment can be used as the parameter t. What happens when t is a rational number?]

   (b) Explain how this same idea permits you to find antiderivatives of all functions of the form $\frac{f(\sin\theta, \cos\theta)}{g(\sin\theta, \cos\theta)}$, where f and g are polynomial functions of two variables.

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