Solution

Math 8 Winter 2019
Midterm II

Your name: ______________________________

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INSTRUCTIONS

This is a closed book, closed notes exam.

You have 2 hours.

There are 10 problems.

Use of calculators is not permitted.

GOOD LUCK!

Free Response. *On each question you must show your work. No credit is given for solutions without supporting calculations. You will get partial credit for partially correct answers.*

(1) *(8 points)* Let L be the line with the following parametric equations:

$$x = 1 + t\ ,\ y = 2 - t \quad \text{and} \quad z = 3 + t.$$

Find the point(s) of the intersection between L and the cylinder $C:\ x^2 + y^2 = 9$.

Solve t such that

$$(1+t)^2 + (\underset{2-t}{\cancel{3+t}})^2 = 9$$

$$\Leftrightarrow \quad t^2 + 2t + 1 + t^2 - 4t + 4 = 9$$

$$\Leftrightarrow \quad 2t^2 - 2t - 4 = 0$$

$$\Leftrightarrow \quad t^2 - t - 2 = 0 \quad \Leftrightarrow \quad (t-2)(t+1) = 0$$

$t = 2$ or -1.

When $t = 2$, $(x, y, z) = (3, 0, 5)$

$t = -1$, $(x, y, z) = (0, 3, 2)$

$(3, 0, 5)$ and $(0, 3, 2)$ are the intersection pts.

(2) *(10 points)* Find the parametric equations for the line of intersection between the planes

$$2x + y + z = 3 \quad \text{and} \quad 3x - y + z = 1.$$

First, find a point on the line.

Set $x = 0$. Then

$$y + z = 3$$

$$-y + z = 1.$$

Add two equations to obtain $2z = 4$, $z = 2$.

Then $y + z = 3$,

$$y = 1$$

$(0, 1, 2)$ is a pointon the plane.

~~Normal vec~~

Parallel vector to the line is

$$\begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 3 & -1 & 1 \end{vmatrix} = \langle 4, 7, -5 \rangle.$$

The line is

$$x = 4t$$

$$y = 1 + 7t$$

$$z = 2 - 5t$$

(3) *(12 points)* Let

$$P = (2,1,0)\ ,\ Q = (3,2,-1) \quad \text{and} \quad R = (-2,0,2)$$

be three points in the space.

a) Find the area of the triangle with vertices P, Q and R.

$\vec{PQ} = \langle 1, +1, -1 \rangle$

$\vec{PR} = \langle -4, -1, 2 \rangle$

$$\text{Area} = \frac{1}{2} | \vec{PQ} \times \vec{PR} |$$

$$= \frac{1}{2} \left| \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ -4 & -1 & 2 \end{vmatrix} \right| = \frac{1}{2} | \langle 1, 2, 3 \rangle | = \frac{1}{2} \sqrt{1^2 + 2^2 + 3^2}$$

$$= \frac{1}{2} \sqrt{14}$$

b) Find the linear equation of the plane containing the triangle PQR.

Use the point P :

$(x-2) + 2(y-1) + 3z = 0$, or

$x + 2y + 3z = 4$

(4) *(6 points)* Given the two vectors

$$\mathbf{u} = \langle 2, 1, -1 \rangle \quad \text{and} \quad \mathbf{v} = \langle 3, -1, 2 \rangle.$$

Find the vector projection $proj_{\mathbf{v}}(\mathbf{u})$ of $\mathbf{u}$ on to $\mathbf{v}$.

$$proj_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \cdot \vec{v}$$

- $\vec{u} \cdot \vec{v} = (2,1,-1) \cdot (3,-1,2)$
  $= 6 - 1 - 2 = 3$

- $|\vec{v}|^2 = 9 + 1 + 4 = 14$

- $proj_{\vec{v}}(\vec{u}) = \boxed{\frac{3}{14}(3,-1,2)}$
  $= \boxed{\left(\frac{9}{14}, \frac{-3}{14}, \frac{6}{14}\right)}$

(5) *(10 points)* Let c be a curve given by

$$c: \mathbf{r}(t) = \left\langle 2t^3, 1+t^2, \frac{1}{2+3t} \right\rangle \quad \text{where } t \in [-3,3], t \neq -\frac{2}{3}.$$

Find the point(s) on the curve c where the tangent vector of the curve is orthogonal to the vector $\mathbf{v} = \langle 1, 3, 0 \rangle$.

Find t, where $r'(t) \cdot \vec{v} = 0$

$$r'(t) = \left(6t^2, 2t, \frac{-3}{(2+3t)^2}\right)$$

$$r'(t) \cdot \vec{v} = 6t^2 + 6t = 0$$

$$\Rightarrow t^2 + t = 0$$

$$\Rightarrow t(t+1) = 0$$

$$\Rightarrow t = 0 \quad \text{or} \quad t = -1$$

$$\boxed{P_1 = \left(0, 1, \frac{1}{2}\right)} = r(0) \qquad \boxed{P_2 = (-2, 2, -1)} = r(-1)$$

$$r'(0) = \left(0, 0, -\frac{3}{2}\right) \perp \vec{v} \qquad r'(-1) = (6, -2, -3) \perp \vec{v}$$

(6) *(10 points)* Find the arclength $\ell(c)$ of the curve
$$c: \ \mathbf{r}(t) = \langle e^t\cos(t), 1, e^t\sin(t)\rangle \quad \text{where} \quad t \in [0,\pi].$$

Find $\int_0^\pi |r'(t)|\,dt = \ell(c)$

$r'(t) = (e^t(\cos(t) - \sin(t)), 0, e^t(\sin(t)+\cos(t)))$

$|r'(t)| = \left(e^{2t}(\cos^2(t) - \cancel{2\cos(t)\sin(t)} + \sin^2(t)) + e^{2t}(\cos^2(t) + \cancel{2\cos(t)\sin(t)} + \sin^2(t))\right)^{1/2}$

$= \left(e^{2t}\cdot 2(\underbrace{\cos^2(t) + \sin^2(t)}_{=1})\right)^{1/2}$

$= e^t \cdot \sqrt{2}$

$\ell(c) = \int_0^\pi e^t\sqrt{2}\,dt = \sqrt{2}(e^\pi - e^0) =$

$\boxed{\sqrt{2}(e^\pi - 1)}$

(7) *(12 points)* Let $\mathbf{r}(t)$ be the space curve which is the curve of intersection of the surfaces

$$z = x + 1 \quad \text{and} \quad x^2 + y^2 + (z-1)^2 = 4.$$

a) Parametrize the curve $\mathbf{r}(t)$.

Plug $z = x+1$ into $x^2+y^2+(z-1)^2 = 4$: $x^2+y^2+(x+1-1)^2 = 2x^2+y^2 = 4$

Put into Standard ellipse form: $\frac{x^2}{2} + \frac{y^2}{4} = \left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

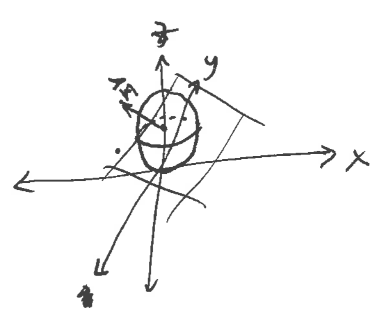
Set $\frac{x}{\sqrt{2}} = \cos t$, $\frac{y}{2} = \sin t$

$\boxed{x = \sqrt{2}\cos t}$, $\boxed{y = 2\sin t}$, $\boxed{z = x+1 = \sqrt{2}\cos t + 1}$.

b) Describe the two surfaces and the curve of intersection $\mathbf{r}(t)$ with your own words and then sketch them.
Note: This part can be solved independently of part a).

$-x + z = 1$ is a plane through $(0,0,1)$ with normal vector $\vec{n} = (-1, 0, 1)$

$x^2+y^2+(z-1)^2 = 4$ is a sphere of radius 2 centered at $(0,0,1)$



The intersection is an ellipse lying on the plane $-x+z=1$.

(8) *(12 points)* Compute the following limits or show that they do not exist.

a) $\lim_{(x,y)\to(0,0)} \frac{y^4 \cdot x}{y^4 + 3y^2}$.

$$\frac{y^4 \cdot x}{y^4+3y^2} = \frac{y^4 \cdot x}{y^2(y^2+3)} = \frac{y^2 \cdot x}{y^2+3}$$

$$\lim_{(x,y)\to(0,0)} \frac{y^2 \cdot x}{y^2+3} = \frac{0^2 \cdot 0}{0+3} = 0/3 = 0.$$

b) $\lim_{(x,y)\to(1,0)} \frac{(x-1)^2}{(x-1)^2 + y^3}$.

Along $(1,t)$

$$\lim_{t\to 0} \frac{(1-1)^2}{(1-1)^2+t^3} = \lim_{t\to 0} \frac{0}{t^3} = 0$$

Along $(t,0)$

$$\lim_{t\to 1} \frac{(t-1)^2}{(t-1)^2+0^3} = \lim_{t\to 1} \frac{(t-1)^2}{(t-1)^2} = 1.$$

Limit DNE.

(9) *(10 points)* Let f be the function given by

$$f(x,y) = e^{xy} + \frac{1}{xy^2}. = e^{xy} + x^{-1}y^{-2}$$

Find the following derivatives.

a) $f_x = \boxed{ye^{xy} - x^{-2}y^{-2}}$

b) $f_y = \boxed{xe^{xy} - 2x^{-1}y^{-3}}$

c) $f_{xy} = (f_x)_y = \frac{d}{dy}\left(ye^{xy} - x^{-2}y^{-2}\right)$

$= \boxed{1e^{xy} + xye^{xy} + 2x^{-2}y^{-3}}$

d) $f_{xx} = (f_x)_x = \frac{d}{dx}\left(ye^{xy} - x^{-2}y^{-2}\right) = \boxed{y^2e^{xy} + 2x^{-3}y^{-2}}$

(10) **Short answer.** *(10 points)* You do not have to show your work. However, if you are not sure of your answer, you might want to explain your reasoning.

Match the function with the description that best describes its level sets.

Note: Some descriptions may be used more than once and others not at all.

Functions.

(1) __C__ $f(x, y) = y + x^2$

(2) __E__ $f(x, y) = e^x$

(3) __C__ $f(x, y) = \ln(y - x^2)$

(4) __F__ $f(x, y) = \frac{1}{x+y}$

(5) __G__ $f(x, y) = \frac{x}{y+1}$

$y + x^2 = k \quad y = k - x^2$

$e^x = k \quad x = \ln k \rightarrow$ vertical lines.

$y - x^2 = e^k$

$y = \frac{1}{k} - x$

$x = ky + k$. These lines are not parallel. $y = -1$ is removed.

Descriptions of level sets.

(A) a family of circles

(B) a family of hyperbolas

(C) a family of parabolas

(D) a family of horizontal lines

(E) a family of vertical lines

(F) a family of diagonal lines

(G) a family of (non-parallel) lines with one point punctured out