Solution

Math 8 Winter 2019
Midterm I

Your name: ______________________________

Instructor (please check the box): ☐ Samuel Lin ☐ Bjoern Muetzel

INSTRUCTIONS

This is a closed book, closed notes exam.

You have 2 hours.

There are 8 free response problems.

Use of calculators is not permitted.

GOOD LUCK!

Free Response. *On each question you must show your work. No credit is given for solutions without supporting calculations. You will get partial credit for partially correct answers.*

(1) *(11 points)* Find the Taylor polynomials $T_3(x)$ for the following functions at the center point $x = a$.

a) $f(x) = 2x^2 - 2x + 1$ at $a = 1$.

$f(1) = 2-2+1 = 1$

$f'(x) = 4x-2$, and $f'(1) = 4-2=2$

$f''(x) = 4$

$f^{(3)}(x) = 0$

$$T_3(x) = f(1) + \frac{f'(1)}{1}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3$$

$$= 1 + 2(x-1) + 2(x-1)^2$$

b) $g(x) = x^2 + \sin(x)$ at $a = \frac{\pi}{2}$.

$g(\frac{\pi}{2}) = (\pi/2)^2 + \sin(\frac{\pi}{2}) = 1 + \frac{\pi^2}{4}$

$g'(x) = 2x + \cos(x)$, $g'(\frac{\pi}{2}) = \pi$

$g''(x) = 2 + (-\sin(x))$, $g''(\frac{\pi}{2}) = 2-1=1$

$g^{(3)}(x) = -\cos x$, $g^{(3)}(\frac{\pi}{2}) = 0$

$$T_3(x) = \left(1+\frac{\pi^2}{4}\right) + \frac{\pi}{1}(x-\pi/2) + \frac{1}{2!}(x-\frac{\pi}{2})^2$$

(2) *(11 points)* Suppose that we want to estimate $\sqrt{1.01}$ by using the Taylor polynomial $T_3(x)$ of

$$f(x) = \sqrt{1+x} \quad \text{at} \quad a = 0.$$

Give an upper bound on the error when estimating $\sqrt{1.01}$ with this polynomial.
Note: You do not have to find $T_3(x)$ for this question.

$f(x) = \sqrt{1+x}$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$

$f^{(3)}(x) = -\frac{1}{4}\left(\frac{-3}{2}\right)(1+x)^{-5/2} = \frac{3}{8}(1+x)^{-5/2}$

$f^{(4)}(x) = \frac{-15}{16}(1+x)^{-7/2}$

Error of the estimate

$= |R_3(0.01)| = \left| \frac{f^{(4)}(c)}{4!}(0.01)^4 \right|$

$= \left| \frac{-15}{16 \cdot 4!}(1+c)^{-7/2}(0.01)^4 \right|$

$\leq \boxed{\frac{15}{16 \times 4 \times 3 \times 2}(0.01)^4}$, as $(1+c)^{-7/2} < 1^{-7/2}$

(3) *(13 points)* Determine whether the following series converge or not and explain the reason. If a series converges, compute its sum.

a) $\sum_{n=3}^{\infty} \frac{n^2+3}{2n^2}$.

$\lim_{n\to\infty} \frac{n^2+3}{2n^2} = \frac{1}{2} \neq 0 \Rightarrow$ Series diverges

b) $\sum_{n=0}^{\infty} \frac{n!}{2^{2n}}$.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{2^{2(n+1)}} \cdot \frac{2^{2n}}{n!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{2^{2n}}{2^{2n+2}} = \lim_{n\to\infty} (n+1)\cdot\frac{1}{4} = \infty > 1$

$\Rightarrow$ Series diverg.

c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{3^n}$. $= \sum_{n=2}^{\infty} \left(\frac{-1}{3}\right)^n$ Convergent geometric series as $\left|\frac{-1}{3}\right| < 1$

$= \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n - \sum_{n=0}^{1} \left(\frac{-1}{3}\right)^n = \frac{1}{1-\frac{-1}{3}} - 1 - \frac{-1}{3}$

$= \frac{3}{4} - 1 + \frac{1}{3} = \frac{-1}{4} + \frac{1}{3} = \frac{-3+4}{12} = 1/12$

(4) *(10 points)* Determine the radius of convergence of the following power series. Explain your steps.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{2^{2n}}$.

$$\lim_{n\to\infty} \left| \frac{(x+2)^{\cancel{n+1}}}{2^{2(n+1)}} \quad \frac{2^{2n}}{(x+2)^{\cancel{n}}} \right|$$

$$= \lim_{n\to\infty} \left| (x+2) \cdot \frac{2^{2n}}{2^{2n+2}} \right| = \lim_{n\to\infty} \left| \frac{(x+2)}{4} \right| = \left| \frac{(x+2)}{4} \right|$$

Hence the radius $= 4$.

b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)x^{2n+1}}{7^n}$.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+2)\, x^{2(n+1)+1}}{7^{n+1}} \cdot \frac{7^n}{(-1)^n (n+1)\, x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| -1 \cdot \frac{n+2}{n+1} \cdot \frac{1}{7} \cdot x^2 \right|$$

$$= \cancel{0}\ \frac{x^2}{7}$$

If $\frac{x^2}{7} < 1$, then $x^2 < 7$, or equivalently,

$|x| < \sqrt{7}$.

Hence the radius of convergence $= \sqrt{7}$.

(5) *(12 points)* Find power series representations centered at 0 for the following functions by manipulating the geometric series and determine the radius of convergence.
Note: Write down your result in sigma or sum notation and additionally the first three non-zero terms of the sum.

a) $\frac{1}{(1+x)^3}$.

We know: $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ for $|y| < \boxed{1 = R}$

Substitute $\Rightarrow \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ for $|-x| = |x| < \boxed{1 = R}$

Differentiate $\Rightarrow -\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^n x^{n-1} \cdot n$ $\boxed{R=1}$

Differentiate $\Rightarrow \frac{2}{(1+x)^3} = \sum_{n=2}^{\infty} (-1)^n x^{n-2} n(n-1)$ $\boxed{R=1}$

$\Rightarrow \frac{1}{(1+x)^3} = \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n n(n-1) x^{n-2} = 1 - 3x + 6x^2 + \ldots$

Radius of convergence: $\boxed{R=1}$

b) $x^2 \ln(1-3x)$.

$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ for $|y| < 1 = R$

$\frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n$ for $|3x| < 1$, so $|x| < \boxed{\frac{1}{3} = R}$

Integrate $\Rightarrow -\frac{1}{3} \ln(1-3x) = \sum_{n=0}^{\infty} 3^n \frac{x^{n+1}}{n+1} + C$ for $|x| < \frac{1}{3}$

$C = 0$ as $\ln(1-3x) = 0 = \sum_{n=0}^{\infty} 3^n \frac{0^{n+1}}{n+1} + C$

$\Rightarrow x^2 \ln(1-3x) = -3x^2 \sum_{n=0}^{\infty} 3^n \frac{x^{n+1}}{n+1}$ $\boxed{R = \frac{1}{3}}$

$= \sum_{n=0}^{\infty} (-1) 3^{n+1} \frac{x^{n+3}}{n+1}$

$= -3x^3 - \frac{9}{2} x^4 - 9x^5 + \ldots$

Radius of convergence $\boxed{R = \frac{1}{3}}$

(6) *(12 points)* Evaluate the following limits using an appropriate power series expansion. Justify your answer.

a) $\lim_{x\to 1} \dfrac{e^{x-1} - 1 - (x-1)}{3(x-1)^2}$.

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} \quad , \quad e^{x-1} = \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

$$= 1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} \dots$$

So $$\frac{e^{x-1} - 1 - (x-1)}{3(x-1)^2} = \frac{\frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \dots}{3(x-1)^2}$$

$$= \frac{1}{6} + \cancel{\frac{(x-1)}{3!\cdot 3}}^{\,0} + \dots$$

$$\lim_{x\to 1} \frac{e^{x-1} - 1 - (x-1)}{3(x-1)} = \boxed{\frac{1}{6}}$$

b) $\lim_{x\to 0} \dfrac{\sin(x^2) + x^2}{2x}$.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} \dots$$

$$\frac{\sin(x^2) + x^2}{2x} = \cancel{x}^{\,0} - \cancel{\frac{x^5}{2\cdot 3!}}^{\,0} + \cancel{\frac{x^9}{2\cdot 5!}}^{\,0} \dots$$

$$\lim_{x\to 0} \frac{\sin(x^2) + x^2}{2x} = \boxed{0}$$

(7) *(9 points)* (Vectors)

a) Find a unit vector **u** in the same direction as the vector with starting point $P = (1, 2, 4)$ and endpoint $Q = (2, 1, 6)$.

$\vec{PQ} = \langle 2-1, 1-2, 6-4 \rangle = \langle 1, -1, 2 \rangle$

$u = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{\langle 1, -1, 2 \rangle}{\sqrt{1^2+(-1)^2+2^2}} = \langle \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \rangle$

b) Find the cosine of the angle between the vectors $\mathbf{v} = (1, 4, -1)$ and $\mathbf{w} = (3, 0, 1)$.

$V \cdot W = |V| \times |W| \cos\theta$

$|V| = \sqrt{1^2+4^2+(-1)^2} = \sqrt{18}$

$|W| = \sqrt{3^2+1^2} = \sqrt{10}$

$\cos\theta = \frac{V \cdot W}{|V| \times |W|} = \frac{3-1}{\sqrt{18}\sqrt{10}} = \frac{2}{\sqrt{180}}$

(8) *(12 points)* **Short answer.** You do not have to show your work. However, if you are not sure of your answer, you might want to explain your reasoning.

Fill in the blanks with the letter that corresponds to the description of the region in $\mathbb{R}^3$ represented by given equation(s) or inequality.

Note: Some descriptions may be used more than once and others not at all.

(a) ___D___ $2 < (x-1)^2 + (y+1)^2 + z^2$
(b) ___G___ $x = 2$ or $x = 5$
(c) ___B___ $x + 3z = 2$ ← note that y is not specified
(d) ___I___ $4 = x^2 + y^2$ and $x = 0$
(e) ___E___ $1 < x < 7$
(f) ___A___ $y = 4$ and $x = z$

Descriptions.
(A) a line
(B) a single plane
(C) a sphere
(D) the whole space with a ball cut out
(E) the region between a pair of parallel planes
(F) a point
(G) two parallel planes
(H) a cylinder
(I) two parallel lines
(J) a circle