Math 8, Winter 2005

Scott Pauls

Dartmouth College, Department of Mathematics

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With Acroread, CTRL-L switch between full screen and window mode

The gradient

The gradient vector contains all of the directional derivative information:

Given a point (x_0, y_0) and a direction vector $\vec{v} =< a, b >$, the directional derivative of $f(x, y)$ in the direction of $\vec{v}$ at (x_0, y_0) is

$$D_{\vec{v}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{v}$$

Maximum ascent

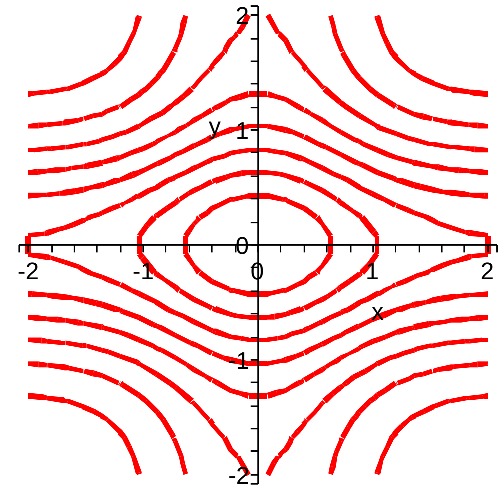
In what direction does a function increase the fastest?

$$
\begin{aligned}
D_{\vec{v}} f &= \nabla f \cdot \vec{v} \\
&= |\nabla f||\vec{v}| \cos(\theta) \\
&= |\nabla f| \cos(\theta)
\end{aligned}
$$

The right hand side is largest when $\cos(\theta) = 1$, i.e. when $\theta = 0$.

- A function f increases most quickly in the direction of the gradient vector, $\frac{\nabla f}{|\nabla f|}$.
- The function decreases most quickly in the direction of $-\frac{\nabla f}{|\nabla f|}$
- The function has directional derivative zero in the directions perpendicular to the gradient.

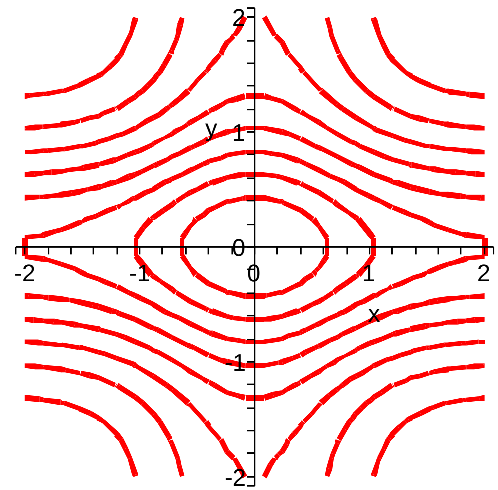
First, we draw a contour plot of a funciton $f(x, y)$.

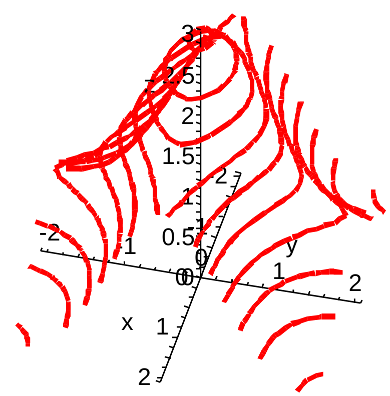


Contour plots and the gradient

First, we draw a contour plot of a funciton $f(x, y)$.

- The contours represent the curves along which f is constant. So the directional derivative in those directions is zero!





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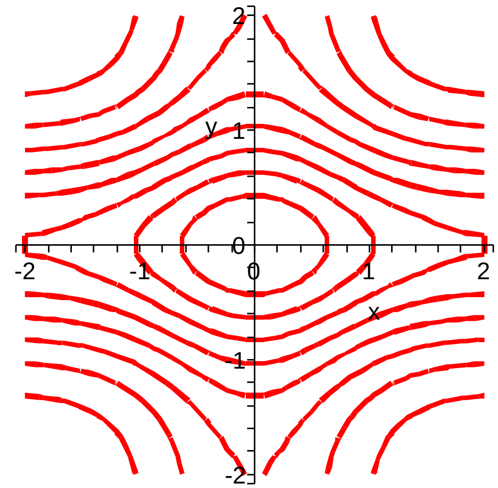
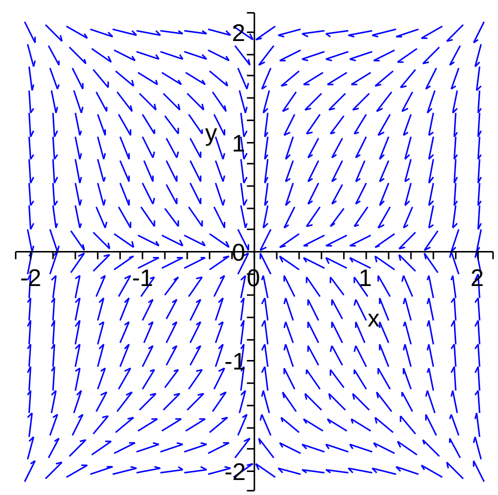


- Draw vectors representing the gradient vector at each point.



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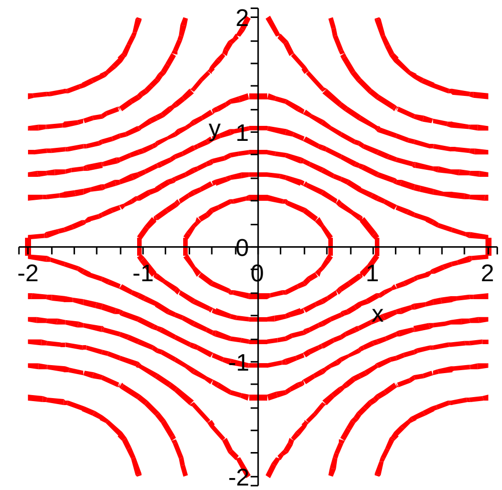
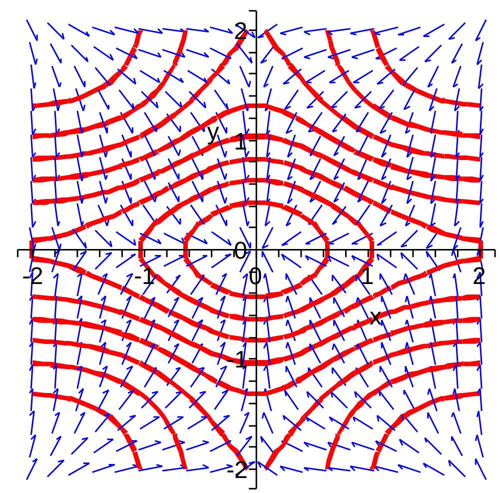
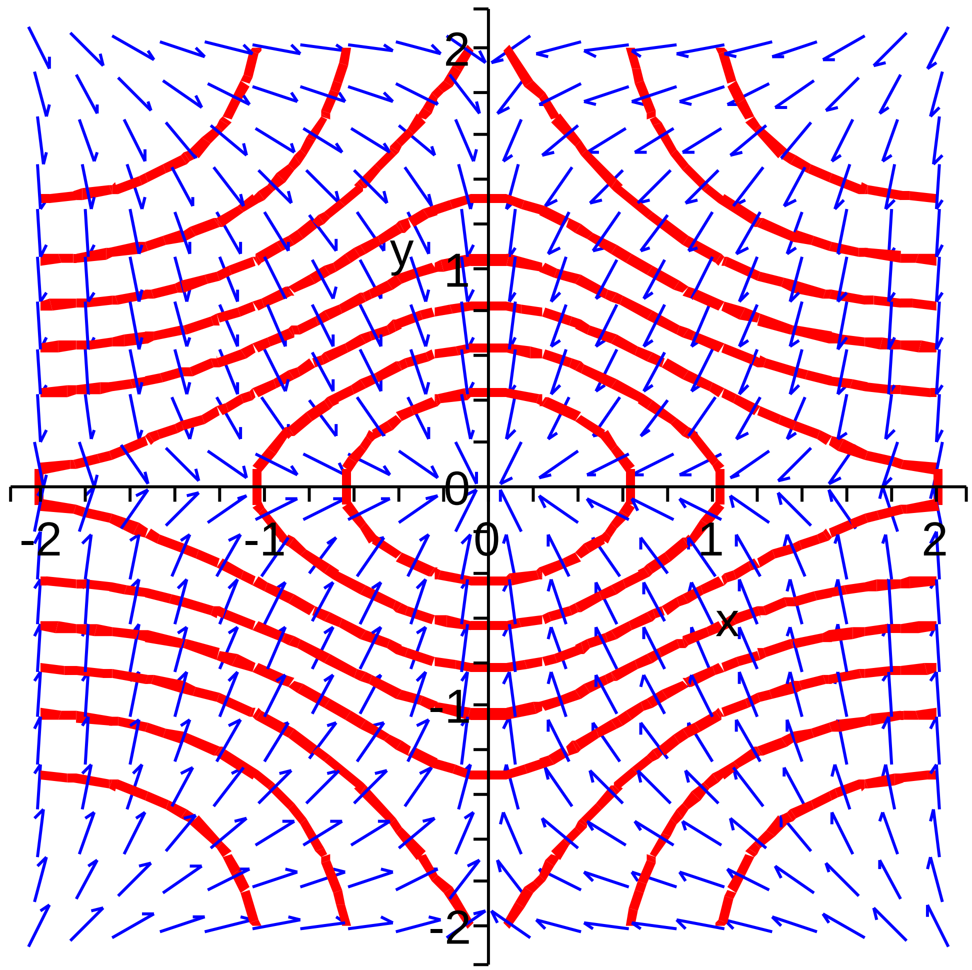


- Draw vectors representing the gradient vector at each point.

- Note the the vectors are perpendicular to the curves



Big picture



A closer look...

Conclusions

- We can reach a (local) maximum of f by following curves that are tangent to the gradient.
- We can reach a (local) minimum of f by following the curves that are tangent to the negative of the gradient.
- Where is a max or min? At a place where the gradient vanishes!

Local extrema

The critical points of a function $f(x, y)$ are the points (x_0, y_0) where

$$\nabla f(x_0, y_0) = 0$$

All local maxima and minima occur at critical points.

Examples:

- $f(x, y) = x^2 + y^2$
- $f(x, y) = x^2 - 2xy + y^4$