Math 8, Winter 2005

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With Acroread, CTRL-L switch between full screen and window mode

One more limit example

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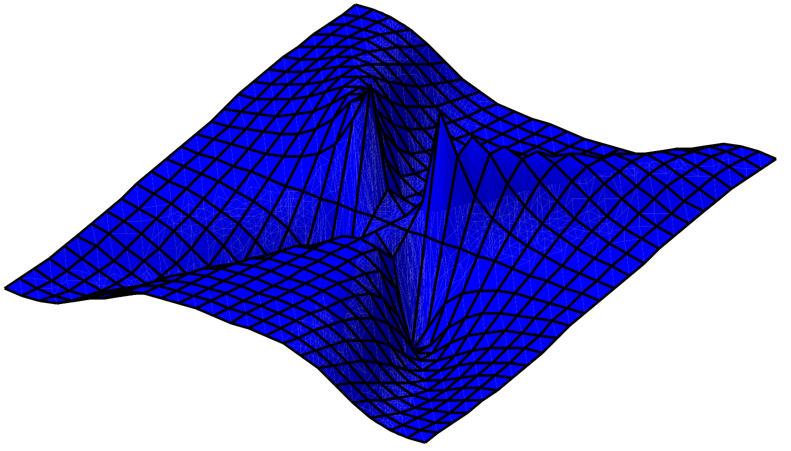

Derivatives

How can we differentiate a function of two variables, $f(x,y)$? We can start our investigation by reducing the problem to one variable. Choose a point $(x_0, y_0, f(x_0, y_0))$ where we want to understand the derivative and a direction $\vec{v} =< a, b >\in \mathbb{R}^2$.

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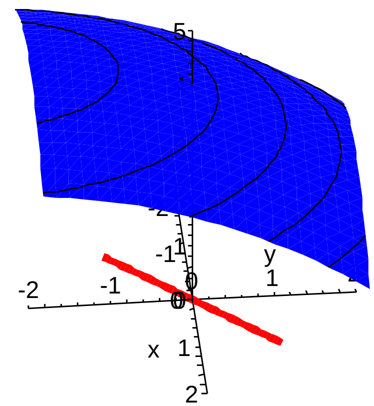

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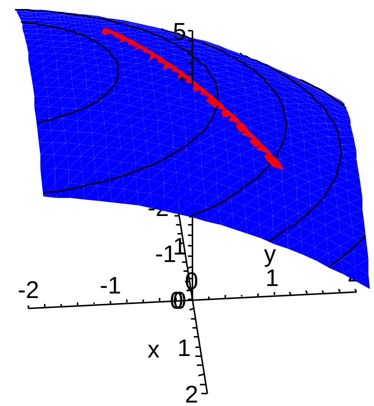

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- Lift this line to the surface

$$\vec{c}_{\vec{v}}(t) =< x_0 + ta, y_0 + tb, f(x_0 + ta, y_0 + tb) >$$

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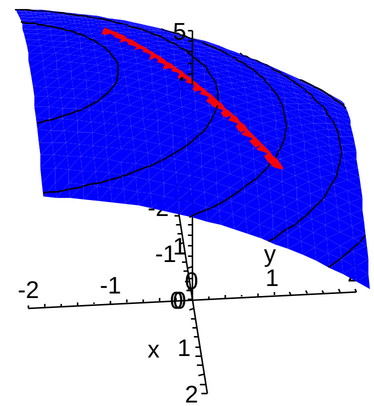

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Take the derivative of the space curve:

$$\vec{c}\,'_{\vec{v}}(0) = \left\langle a, b, \left(\frac{d}{dt} f(x_0 + ta, y_0 + tb)\right)_{t=0} \right\rangle$$

The last coordinate is called the *directional derivative* of f at (x_0, y_0) in the direction of $\vec{v}$, $D_{\vec{v}} f(x_0,y_0) = \frac{d}{dt}|_{t=0} f(x_0 + ta, y_0 + tb)$.

Directional Derivatives

Notice: we have one derivative for each direction vector $\vec{v}$. The directional derivative is a measure of the rate of change of the function in that direction. If we let $\vec{v}$ be one of $\{\vec{i}, \vec{j}\}$ we get special directional derivatives:

$$D_{\vec{i}} f = \frac{\partial}{\partial x} f = f_x$$

$$D_{\vec{j}} f = \frac{\partial}{\partial y} f = f_y$$

The are called the *partial derivatives* of f with respect to x and y. Equivalent definition:

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Higher order derivatives

Compute higher order derivatives by iterating the derivative process.

Examples:

- $f(x,y) = x^2 + \sin(xy)$
- $f(x,y) = \frac{xy}{x^2+y^2}$

Functions of several variables

Given $f(x, y, z, w)$, compute partial derivatives in exactly the same way: hold all variables but one constant and differentiate in the remaining variable.

Examples:

- $x^2 + y^3 + xyz$
- $\tan(x + 2y + 3z + 4w)$

The gradient

We collect all partial derivatives into a derivative vector:

$$\nabla f = < f_x, f_y, f_z >$$

∇f is called the gradient of the function f.

Application: tangent plane to $z - f(x, y) = 0$.

- f_x and f_y give two tangent directions on the surface

$$\vec{v} = < 1, 0, f_x >$$

$$\vec{w} = < 0, 1, f_y >$$

- $$\vec{n} = \vec{v} \times \vec{w} = < -f_x, -f_y, 1 >$$

Tangent planes

The tangent plane to $F(x, y, z) = z - f(x, y) = 0$ is given by:

$$\vec{n} \cdot < x, y, z > - < x_0, y_0, z_0 >= 0$$

or

$$\nabla F \cdot < x - x_0, y - y_0, z - z_0 >= 0$$