

## Math 8, Winter 2005

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# Power Series

There are several ways to make new power series out of old ones:

Integration:

$$\int \sum_{n=0}^{\infty} c_n (x - a)^n dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x - a)^{n+1}$$

Differentiation:

$$\frac{d}{dx} \sum_{n=0}^{\infty} c_n (x - a)^n dx = \sum_{n=1}^{\infty} n c_n (x - a)^{n-1}$$

Substitution: for example, let  $x = \alpha u + \beta$

$$\sum_{n=0}^{\infty} c_n (x - a)^n dx = \sum_{n=0}^{\infty} c_n (\alpha u + \beta - a)^n = \sum_{n=0}^{\infty} c_n \alpha^n \left( u - \frac{\beta - a}{\alpha} \right)^n$$



# Power Series

Integration and Differentiation do not change the radius of convergence, but substitution may!

Suppose  $\sum_{n=0}^{\infty} c_n (x - a)^n$  has radius of convergence  $R$ , i.e.

$$\lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right| = R$$

Then, after substituting  $x = \alpha u + \beta$ , we have the series:

$$\sum_{n=0}^{\infty} c_n \alpha^n \left( u - \frac{\beta - a}{\alpha} \right)^n$$

Performing the ratio test yields:

$$\lim_{n \rightarrow \infty} \left| \frac{c_{n+1} \alpha^{n+1} \left( u - \frac{\beta - a}{\alpha} \right)^{n+1}}{c_n \alpha^n \left( u - \frac{\beta - a}{\alpha} \right)^n} \right| < 1$$

$$\lim_{n \rightarrow \infty} \left| \frac{c_{n+1} \alpha}{c_n} \right| \left| \left( u - \frac{\beta - a}{\alpha} \right) \right| < 1$$



# Substitution (con't)

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}\alpha}{c_n} \right| \left| \left( u - \frac{\beta - a}{\alpha} \right) \right| &< 1 \\ \left| \left( u - \frac{\beta - a}{\alpha} \right) \right| &< \frac{1}{\alpha} \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right| \\ &= \frac{R}{\alpha} \end{aligned}$$

So the radius of convergence of the new series after substitution is  $\frac{R}{\alpha}$ .



Power series define functions on their intervals of convergence. Can we find a different description for the function?

$$\sum_{n=0}^{\infty} c_n (x - a)^n = f(x)$$

Or, given a function, can we find a power series representation?

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$



Example: geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Use this to find a power series representation for

•

$$f(x) = \frac{1}{(1-x)^2}$$

•

$$f(x) = \ln(1-x)$$

•

$$f(x) = \frac{1}{1+2x^2}$$



What about a function like

$$f(x) = \sin(x)$$

Idea:

- Goal: a power series representation about  $a$
- The  $m^{\text{th}}$  partial sum of a series

$$\sum_{n=0}^{\infty} c_n (x - a)^n$$

is a polynomial of degree  $m$ :

$$s_m = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_m(x - a)^m$$

- Find polynomials that closely match the function,  $f(x)$ , near  $x = a$ .



# $m = 0$ and $m = 1$

$$m = 0$$

$s_0 = c_0$  so simply pick  $c_0 = f(a)$ .

$$m = 1$$

$$s_1 = c_0 + c_1(x - a) = f(a) + c_1(x - a)$$

Find  $c_1$  so that  $f(x) - (f(a) + c_1(x - a))$  is as small as possible near  $x = a$ :

$$f(x) - (f(a) + c_1(x - a)) = (f(x) - f(a)) + c_1(x - a)$$

Dividing by  $(x - a)$  gives:

$$\frac{(f(x) - f(a))}{x - a} + c_1 \sim f'(a) + c_1$$

when  $x$  is close to  $a$ . So pick  $c_1 = f'(a)$ .





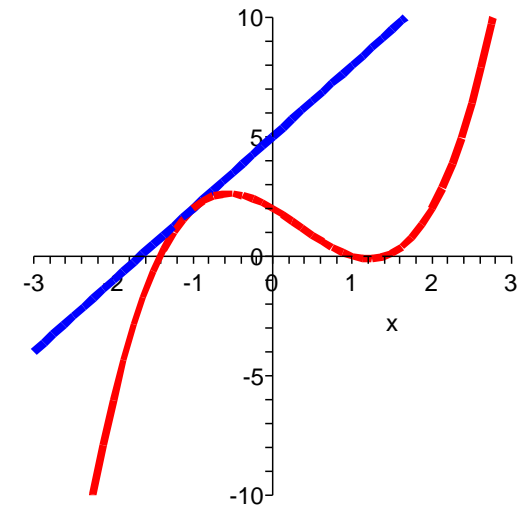
# $m = 1$ geometrically

Using these values, what is  $s_1 = f(a) + f'(a)(x - a)$ ?



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# $m = 1$ geometrically



Using these values, what is  $s_1 = f(a) + f'(a)(x - a)$ ?

It is just the tangent line to  $f$  at  $x = a$ !

