1. (15) (Short Answer) Compute the following, if they exist. If not, explain why.

(i) Find the area of the parallelogram whose vertices are $(0,0,0)$, $(1,3,1)$, $(3,1,1)$, and $(4,4,2)$.

Answer: Area $= |\langle 1,3,1\rangle \times \langle 3,1,1\rangle| = |\langle 2,2,-8\rangle| = \sqrt{4+4+64} = \sqrt{72}$

(ii) Let $\vec{\mathbf{r}}(t)$ be the vector-valued function $\vec{\mathbf{r}}(t) = \langle t^2, e^t, \cos(t)\rangle$.
Find $\lim\limits_{h\to 0} \frac{\vec{\mathbf{r}}(h)-\vec{\mathbf{r}}(0)}{h} =$ _______________.

Answer: $\vec{\mathbf{r}}'(t) = \langle 2t, e^t, -\sin(t)\rangle$.
$\lim\limits_{h\to 0} \frac{\vec{\mathbf{r}}(h)-\vec{\mathbf{r}}(0)}{h} = \vec{\mathbf{r}}'(0) = \langle 0,1,0\rangle$.

(iii) Find the distance between the point $(4,-3,2)$ and the plane $x+y+z=0$.

Answer:

$$D = \frac{|4-3+2|}{\sqrt{1^2+1^2+1^2}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

(iv) The arc length L of the curve defined parametrically by

$$\langle x,y,z\rangle = \langle 13\cos(t), 5\sin(t), 12\sin(t)\rangle, \quad 0 \le t \le 2\pi,$$

is given by $L =$ _______________.

Answer: $\vec{\mathbf{r}}'(t) = \langle -13\sin(t), 5\cos(t), 12\cos(t)\rangle$. $|\vec{\mathbf{r}}'(t)| = \sqrt{169} = 13$.

$$L = \int_0^{2\pi} |\vec{\mathbf{r}}'(t)|\,dt = \int_0^{2\pi} 13\,dt = 26\pi$$

(v) If $z = \ln(\sqrt{x^2+y^2})$ then $\frac{\partial z}{\partial x} =$ _______________.

Answer: $z = \ln(x^2+y^2)^{\frac{1}{2}} = \frac{1}{2}\ln(x^2+y^2)$.

$$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$$

2. (10) (True or False) Circle the correct answer for each of the following statements. Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ denote arbitrary non-zero vectors.

False (i) The cross product of two unit vectors is also a unit vector.

True (ii) Two distinct planes in $\mathbb{R}^3$ that do not intersect in a line must be parallel.

False (iii) If line L_1 is skew to line L_2 and L_2 is skew to line L_3 then L_1 must be skew to L_3.

True (iv) The vectors $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ and $\vec{\mathbf{u}} \times 3\vec{\mathbf{v}}$ are parallel.

True (v) If $\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = -1$ and $\vec{\mathbf{u}} \bullet \vec{\mathbf{w}} = 3$ then $\vec{\mathbf{u}}$ is perpendicular to the vector $6\vec{\mathbf{v}} + 2\vec{\mathbf{w}}$.

False (vi) The normal vector to the plane $z = 3x - 7y + 2$ is the vector $\vec{\mathbf{n}} = \langle 3, -7, 2 \rangle$.

True (vii) The direction vector of the line

$$\frac{3-x}{2} = \frac{y-5}{1} = \frac{z-2}{3}$$

is the vector $\vec{\mathbf{v}} = -2\vec{\mathbf{i}} + \vec{\mathbf{j}} + 3\vec{\mathbf{k}}$.

True (viii) The planes $x - y + z = 0$ and $x + 2y + 3z = 6$ intersect.

True (ix) If a force $\vec{\mathbf{F}}$ is applied to a vector $\vec{\mathbf{v}}$ then the magnitude of the torque is $|\vec{\mathbf{F}}||\vec{\mathbf{v}}|\sin(\theta)$, where θ is the angle between $\vec{\mathbf{v}}$ and $\vec{\mathbf{F}}$.

False (x) The mixed partial derivatives f_{xy} and f_{yx}, if they exist, of a function $f(x, y)$ are equal: $f_{xy} = f_{yx}$.

3. (20) (Multiple Choice) Circle the correct answer for each of the following. No work need be shown for credit.

(i) The plane passing through the points $(0,0,0)$, $(2,-4,6)$, and $(5,1,3)$ is given by

A. $\langle x,y,z\rangle \times \langle 3,5,-3\rangle = 0$

B. $\langle x,y,z\rangle \bullet \langle 3,5,-3\rangle = 0$

(C.) $\underline{\langle x,y,z\rangle \bullet [\langle 5,1,3\rangle \times \langle 2,-4,6\rangle] = 0}$

D. $\langle x,y,z\rangle \times [\langle 5,1,3\rangle \bullet \langle 2,-4,6\rangle] = 0$

(ii) Which one of the following expressions does NOT describe a line passing through the origin?

A. $\langle x,y,z\rangle = \langle 2t,t,-6t\rangle$

(B.) $\underline{x+y+z=0}$

C. $\frac{x-1}{2} = \frac{y-2}{4} = \frac{z+1}{-2}$

D. $x = 5t, y = \frac{3}{2}t, z = -\frac{3}{2}t$

(iii) A drug manufacturer needs to purchase 4 kg of aluminum phosphate, 2 kg of benzalkonium chloride, and 1 kg of cromolyn sodium. These chemicals cost \$250/kg, \$86/kg,and \$3061/kg respectively. Let $\vec{\mathbf{v}} = \langle 4,2,1\rangle$ annd $\vec{\mathbf{w}} = \langle 250,86,3061\rangle$. Which of the following is the total cost of this purchase?

A. The cross product of $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$.

B. The triple product of $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$.

(C.) <u>The dot product of $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$</u>.

D. The cross product of $\vec{\mathbf{w}}$ and $\vec{\mathbf{v}}$

(iv) The level curves (at height k above the xy-plane) for the function $z = \sqrt{x^2+y^2}$ are

A. lines $y = (1+k)x$ where $0 \le k < \infty$.

(B.) <u>circles of radius k centered at the origin</u>.

C. parabolas $y = kx^2$ through the origin.

D. circles of radius k^2 centered at the origin.

(v) The domain of the function

$$f(x,y) = \sqrt{x^2+y^2-1} + \ln(4-x^2-y^2)$$

is equal to the set of all points (x,y) in the plane whose distance r to the origin satisfies:

(A.) $\underline{1 \le r < 2}$

B. $1 < r < 4$

C. $1 < r < 2$

D. $1 < r \le 4$

4. (20) Consider the following two planes in space:

$$\mathcal{P}_1 : x + y - z = 0$$

$$\mathcal{P}_2 : x - 3y + z = 0$$

Please show your work and simplify answers, where possible, in order to receive full credit.

(i) What are the normal vectors $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$ of planes $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively?

Answer: $\vec{\mathbf{n}}_1 = \langle 1, 1, -1 \rangle$ and $\vec{\mathbf{n}}_2 = \langle 1, -3, 1 \rangle$.

(ii) Find the cosine of the angle between the two planes.

Answer: $\cos(\theta) = \sqrt{\frac{3}{11}}$.

(iii) Find the cross product $\vec{\mathbf{n}}_1 \times \vec{\mathbf{n}}_2$.

Answer: $\vec{\mathbf{n}}_1 \times \vec{\mathbf{n}}_2 = \langle -2, -2, -4 \rangle$.

(iv) Find a point $P(a, b, c)$ on the line of intersection of these two planes whose z coordinate is $c = 4$.

Answer: $P(2, 2, 4)$.

(v) Find the parametric equations of the line of intersection of planes $\mathcal{P}_1$ and $\mathcal{P}_2$.

Answer: $x = 2 - 2t, y = 2 - 2t, z = 4 - 4t$.

5. (15) An electron is spiralling in a magnetic field. It's velocity vector at time t is given by

$$\vec{\mathbf{v}}(t) = \langle 2t, -3\sin(t), -3\cos(t) \rangle$$

where time t is in seconds and distance is in meters. It begins moving at time $t = 0$ from the point $P(-2, 3, 0)$. Please show your work and simplify answers, where possible, in order to receive full credit.

(i) At what time is the speed of the electron 5 meters/second?

Answer: Solve $|\vec{\mathbf{v}}(t)| = \sqrt{4t^2 + 9} = 5 \implies t = 2$ seconds.

(ii) What is the acceleration vector $\vec{\mathbf{a}}(t)$ of the electron?

Answer: $\vec{\mathbf{a}}(t) = \langle 2, -3\cos(t), 3\sin(t) \rangle$

(iii) Find the position vector $\vec{\mathbf{r}}(t)$ of the electron.

Answer: $\vec{\mathbf{r}}(t) = \langle t^2 - 2, 3\cos(t), -3\sin(t) \rangle$

(iv) The magnetic field vector at time t is given by

$$\vec{\mathbf{F}}(t) = \langle t - 2, 2\sin(t), 2\cos(t) \rangle.$$

At what time(s) t is the electron moving perpendicular to the magnetic field vector?

Answer: Solve $\vec{\mathbf{v}} \bullet \vec{\mathbf{F}} = 2t^2 - 4t - 6 = 2(t-3)(t+1) = 0$.

The electron is moving perpendicular to the magnetic field vector at time $t = 3$ seconds.

6. (20) Compute the following limits (if they exist) concerning the function

$$f(x,y) = \frac{xy^3}{x^2 + 4y^6}.$$

Where appropriate, write DNE if the limit "does not exist" (and explain why) or NEIG if "not enough information is given" to compute the limit.

(i) The limit of $f(x,y)$ as (x,y) approaches the origin along a line of slope m.

Answer: Along the line $y = mx$ we have $f(x,mx) = \frac{m^3x^4}{x^2+4m^6x^6}$.

$$\lim_{(x,y)\to(0,0)} f(x,mx) = \lim_{x\to 0} \frac{m^3x^4}{x^2 + 4m^6x^6} = 0$$

(ii) The limit of $f(x,y)$ as (x,y) approaches the origin along the curve $y = k\sqrt{x}$, where $k > 0$.

Answer: Along the curve $y = k\sqrt{x}$ we have $f(x,k\sqrt{x}) = \frac{k^3x^{5/2}}{x^2+4k^6x^3}$.

$$\lim_{(x,y)\to(0,0)} f(x,k\sqrt{x}) = \lim_{x\to 0} \frac{k^3x^{5/2}}{x^2 + 4k^6x^3} = 0$$

(iii) The limit of $f(x,y)$ as (x,y) approaches the origin along the curve $y = C\sqrt[3]{x}$ where $C \neq 0$.

Answer: Along the curve $y = C\sqrt[3]{x}$ we have $f(x,C\sqrt[3]{x}) = \frac{C^3}{1+4C^6}$.

$$\lim_{(x,y)\to(0,0)} f(x,C\sqrt[3]{x}) = \lim_{x\to 0} \frac{C^3}{1+4C^6} = \frac{C^3}{1+4C^6} \neq 0$$

(iv) What is $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$?

Answer: DNE. The limit does not exist because the limit in part (iii) differs from the limits in (i) and (ii).

(v) What is $\displaystyle\lim_{(x,y)\to(1,-1)} f(x,y)$?

Answer: Since $f(x,y)$ is continuous at $(1,-1)$ we have

$$\lim_{(x,y)\to(1,-1)} f(x,y) = f(1,-1) = -\frac{1}{5}$$