Solution

Math 8 Spring 2018
Midterm II

Your name: ______________________________

Instructor (please circle): ☐ Vardayani Ratti ☐ Bjoern Muetzel

INSTRUCTIONS

Except on problems that specify short answer, you must show your work in order to receive credit.

You have 2 hours.

This is a closed book, closed notes exam. Use of calculators is not permitted.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Good luck!

exercise 1. (10 points)

a) Determine the angle between the planes

$$E_1 : -x + y - z = -4 \quad \text{and} \quad 3x - 2y - z = 5. : E_2$$

$\vec{n}_1 = (-1, 1, -1)$ $\vec{n}_2 = (3, -2, -1)$

(normal vectors of E_1 and E_2)

$\Rightarrow \measuredangle (E_1, E_2) = \alpha$

$\cos(\alpha) = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = -\frac{4}{\sqrt{3} \cdot \sqrt{14}}$

$\alpha = \arccos\left(-\frac{4}{\sqrt{42}}\right)$ (angle between E_1 and E_2)

b) Find an equation of the plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$.

We know: 1.) $P = (1, -1, -1)$ in E

2.) normal vector $\vec{n}$ of E is $\vec{n} = (5, -1, -1)$

$\Rightarrow E : \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} - P \right\rangle \cdot \vec{n} = 0$

$5(x-1) + (-1)(y+1) + (-1)(z+1) = 0$

or $5x - y - z = 7$

exercise 2. (8 points)

a) Find the parametric equation for the line that intersects the x-axis at $x = 5$ and goes through the point $Q = (3, 4, -8)$.

- $L \cap$ x-axis at $x = 5$

$\Rightarrow P = (5, 0, 0)$ in L

- $Q = (3, 4, -8)$ in L

$L: \; P + t(Q - P) \qquad t \text{ in } \mathbb{R}.$

$(5, 0, 0) + t(-2, 4, -8) \qquad t \text{ in } \mathbb{R}$

parametric equation:

$x = 5 - 2t \quad y = 4t \quad z = -8t$

b) Is the point $(-1, 28, -56)$ on the line?

If the point is on the line, then

I) $5 - 2t = -1 \Rightarrow t = 3$ # a contradiction

II) $4t = 28 \Rightarrow t = 7$

III) $-8t = 56$

Hence P is not on the line L.

exercise 3. (10 points) Compute the following limits or show that they do not exist.

a)

$$\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{3(x^2+y^2)}.$$

Set $r = x^2+y^2$. Then $r=0 \Leftrightarrow (x,y)=(0,0)$

$$\lim_{r\to 0} \frac{e^r-1}{3r} = \lim_{r\to 0} \frac{\left(1+r+\frac{r^2}{2!}+\frac{r^3}{3!}+\dots\right)-1}{3r}$$

$$= \lim_{r\to 0} \frac{1}{3} + \underbrace{\frac{r}{6}}_{0} + \underbrace{\frac{r^2}{3\cdot 3!}}_{0} + \underbrace{\dots}_{0} \dots = \boxed{\frac{1}{3}}$$

b)

$$\lim_{(x,y)\to(1,0)} \frac{4y^2}{3y^2+(x-1)^4}.$$

For $r_1(t) = (1,0)+t(1,0) = (1+t,0)$ $t \in \mathbb{R}$

$r_2(t) = (1,0)+t(0,1) = (1,t)$ $t \in \mathbb{R}$

we have: $r_1(0) = r_2(0) = (1,0)$

$$\lim_{t\to 0} f(r_1(t)) = \lim_{t\to 0} \frac{0}{t^4} = \boxed{0}$$

$$\neq$$

$$\lim_{t\to 0} f(r_2(t)) = \lim_{t\to 0} \frac{4t^2}{3t^2} = \boxed{\frac{4}{3}}$$

Hence the limit does not exist.

exercise 4. (10 points) Consider the two surfaces

$$S_1 : \ y = z^3 \quad \text{and} \quad S_2 : \ x^2 + z^2 = 4.$$

a) Parametrize the curve $\mathbf{r}(t)$ which is the curve of intersection of the surfaces S_1 and S_2.

S_2 can be parametrized by

$(2\cos(t), x, 2\sin(t))$ t in $\mathbb{R}$, y in $\mathbb{R}$

Check: $(\underbrace{2\cos(t)}_{=x})^2 + (\underbrace{2\sin(t)}_{=z})^2 = 4$

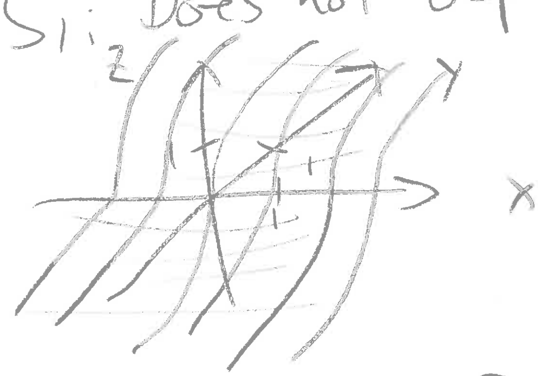
To satisfy the equation for S_1 we set

$y = z^3 = 8\sin^3(t)$. Hence

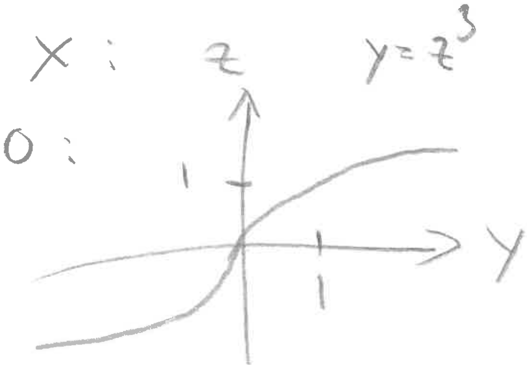
$S_1 \cap S_2$: $(2\cos(t), 8\sin^3(t), 2\sin(t))$ t in $\mathbb{R}$

b) **Short answer, no justification needed.** Give a description or sketch each of the two surfaces in part a).

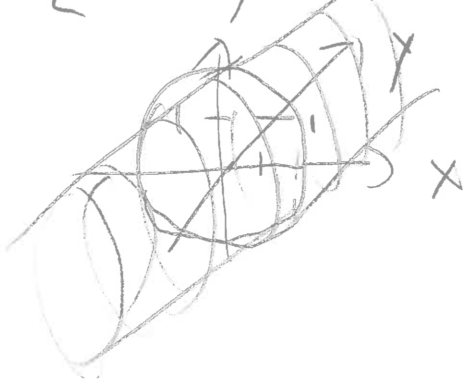
S_1: Does not depend on x:



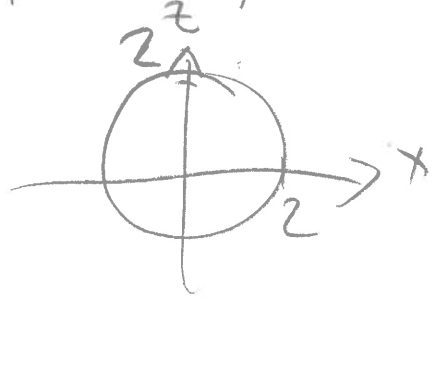
$x=0$:



S_2: Cylinder: Does not depend on y



$y=0$



exercise 5. (10 points) Find the arclength $\ell(c)$ of the curve

$$c:\ \mathbf{r}(t) = \left\langle \ln(t), 2\sqrt{2}t^{1/2}, t \right\rangle \quad \text{where } 1 \le t \le 4.$$

$$\ell(c) = \int_1^4 |r'(t)|\, dt$$

$$r'(t) = \left(\frac{1}{t}, \frac{\sqrt{2}}{t^{1/2}}, 1 \right)$$

$$|r'(t)| = \left(\frac{1}{t^2} + \frac{2}{t} + 1 \right)^{1/2}$$

$$= \left(\frac{1+2t+t^2}{t^2} \right)^{1/2}$$

$$= \left(\left(\frac{1+t}{t} \right)^2 \right)^{1/2} = \frac{1+t}{t} \quad \text{as } t>0$$

$$\ell(c) = \int_1^4 \frac{1+t}{t}\, dt = \int_1^4 \frac{1}{t} + 1\, dt$$

$$= \ln(t) + t \Big|_1^4$$

$$= \ln(4) + 4 - (\underbrace{\ln(1)}_{=0} + 1)$$

$$= \boxed{3 + \ln(4)}$$

exercise 6. (12 points) Find an equation of the plane that passes through the line of intersection L of the planes

$$E_1 : 2x - z = 1 \quad \text{and} \quad E_2 : 4y - x + z = 3$$

and is perpendicular to the plane $E_3 : x + y + z = 1$.

1.) Points on L:

Set $x = 0$: $\left.\begin{array}{l} \text{I}) \; -z = 1 \\ \text{II}) \; 4y + z = 3 \end{array}\right\}$ $\begin{array}{l} z = -1 \\ y = 1 \end{array}$

$P = (0, 1, -1)$

Set $\boxed{z = 0}$ $\left.\begin{array}{l} \text{I}) \; 2x = 1 \\ \text{II}) \; 4y - x = 3 \end{array}\right\}$ $\begin{array}{l} x = \frac{1}{2} \\ 4y = \frac{7}{2} \\ y = \frac{7}{8} \end{array}$

$Q = \left(\frac{1}{2}, \frac{7}{8}, 0\right)$

$\overrightarrow{PQ} = \left(\frac{1}{2}, -\frac{1}{8}, 1\right)$

$\Rightarrow L$: $P + t\left(\frac{1}{2}, -\frac{1}{8}, 1\right)$ or

$P + t\,(4, -1, 8)$ $\qquad t \in \mathbb{R}$

2.) Normal vector of E: $\vec{n}$

a) $\vec{n} \perp (4, -1, 8)$, $\vec{n} \perp \vec{n}_3 = (1, 1, 1)$

$\Rightarrow \vec{n} = (4, -1, 8) \times (1, 1, 1)$: $\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 4 & -1 & 8 \end{vmatrix} = (9, -4, -5)$

$\Rightarrow E$: $\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - P\right) \cdot \vec{n} = 0$:

$9x - 4(y-1) - 5(z+1) = 0$

or $\quad 9x - 4y - 5z = 1$

exercise 7. (10 points) Let c be a curve given by

$$c: \ \mathbf{r}(t) = \left\langle -\frac{1}{t}, t^3, (2t+1)^3 \right\rangle \quad \text{where } \ t \text{ in } \mathbb{R}\setminus\{0\}.$$

Find a parametrization of the tangent line of the curve at the point $P = (-2, \frac{1}{8}, 8)$.

1.) $r'(t) = \left(\frac{1}{t^2}, 3t^2, 6(2t+1)^2\right)$

2.) $P = \left(-2, \frac{1}{8}, 8\right) = \left(-\frac{1}{t}, t^3, (2t+1)^3\right)$

$\Rightarrow \boxed{t = \frac{1}{2}}$

3.) $L: P + t\vec{v} = P + t \cdot r'\left(\frac{1}{2}\right) \quad t \text{ in } \mathbb{R}$

$L: \left(-2, \frac{1}{8}, 8\right) + t \cdot \left(4, \frac{3}{4}, 24\right) \quad t \text{ in } \mathbb{R}$

(tangent line L at P)

exercise 8. (10 points) If $\mathbf{v} = \langle 2, -2, -1 \rangle$, find a vector $\mathbf{u}$ such that $\text{comp}_{\mathbf{v}}(\mathbf{u}) = 1$.

We have:

$$\frac{u \cdot v}{|v|} = 1$$

So $$\frac{(2,-2,-1) \cdot (u_1, u_2, u_3)}{\sqrt{9}} = 1$$

or $$2u_1 - 2u_2 - u_3 = 3$$

Solution: for example:

$$u = (1, 1, -3)$$

exercise 9. (10 points) **Short answer.** You do not have to show your work. However, if you are not sure of your answer, you might want to explain your reasoning.

Match the function with the description of its contour map.

Note: Some descriptions may be used more than once and others might not be used at all.

Functions.

(1) F $f(x, y) = \tan(x + y)$

(2) D $f(x, y) = (1 + y)^2$

(3) C $f(x, y) = \ln(y + x^2)$

(4) B $f(x, y) = \frac{2}{x(y+1)}$

(5) A $f(x, y) = \exp(x^2 + y^2)$

Descriptions.

(A) a family of circles

(B) a family of hyperbolas

(C) a family of parabolas

(D) a family of horizontal lines

(E) a family of vertical lines

(F) a family of diagonal lines

(G) a family of (non-parallel) lines with one point punctured out

1.) $k = \tan(x+y) \Rightarrow \arctan(k) - x = y$

2.) $k = (1+y)^2 \Rightarrow \sqrt{k} - 1 = y$

3.) $k = \ln(y + x^2) \Rightarrow e^k - x^2 = y$

4.) $k = 2/x(y+1) \Rightarrow \frac{2}{xk} - 1 = y$

5.) $k = e^{x^2+y^2} \Rightarrow \ln(k) = x^2+y^2$

exercise 10. (10 points) Let $\mathbf{v} = \langle 2, 1, 2\rangle$ and suppose that $\mathbf{u}$ is a vector of length seven making an angle of $\frac{\pi}{3}$ with $\mathbf{v}$. Evaluate $\mathrm{proj}_{\mathbf{v}}(\mathbf{u})$.

Note: Do not try to find $\mathbf{u}$.

We know:

$$\mathrm{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\cdot\vec{V}$$

$$= \underbrace{\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|\cdot|\vec{u}|}}_{=\cos(\measuredangle(\vec{u},\vec{v}))}\cdot|\vec{u}|\cdot\frac{\vec{v}}{|\vec{v}|}$$

So 1.) $\cos(\measuredangle(\vec{u},\vec{v})) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

2.) $|\vec{u}| = 7$

3.) $|\vec{v}| = \sqrt{9} = 3$

$$\Rightarrow \mathrm{proj}_{\vec{v}}(\vec{u}) = \frac{1}{2}\cdot\frac{7}{3}\cdot\vec{v}$$

$$= \frac{7}{6}(2,1,2) = \left(\frac{7}{3},\frac{7}{6},\frac{7}{3}\right)$$