Worksheet

(1) Let $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{c} = -5\mathbf{j}$. Find the following:
(a) $2\mathbf{a} - 4\mathbf{b}$ (b) $\mathbf{a} \cdot \mathbf{b}$ (c) $|\mathbf{a}|\mathbf{c} \cdot \mathbf{a}$
Solution:
(a) $2\mathbf{a} - 4\mathbf{b} =< -12, 18 >$
(b) $\mathbf{a} \cdot \mathbf{b} = -2(2) - 3(3) = -13$
(c) $|\mathbf{a}| = \sqrt{4+9} = \sqrt{13}$ $\quad \mathbf{c} \cdot \mathbf{a} = -2(0) - 15 = -15$ $|\mathbf{a}|\mathbf{c} \cdot \mathbf{a} = -15\sqrt{13}$

(2) Show that the vectors $< 6, 3 >$ and $< -1, 2 >$ are perpendicular.
Solution:

$$\cos\theta = \frac{< 6,3 > \cdot < -1,2 >}{| < 6,3 > || < -1,2 > |} = \frac{-6+6}{\sqrt{(36+9)(5)}} = 0$$

The angle between the two vectors is $\frac{\pi}{2}$. Thus they are perpendicular.

(3) Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$ where $\mathbf{a} =< 1, 1, 1 >$ and $\mathbf{b} =< 1, -1, 1 >$.
Solution:

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{1-1+1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} = \frac{1}{3} < 1, 1, 1 >$$

$$\text{orth}_{\mathbf{a}}\mathbf{b} = \mathbf{b} - \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} =< 1, -1, 1 > -\frac{1}{3} < 1, 1, 1 >=< 2/3, -4/3, 2/3 >$$

(4) Let $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$, and $\mathbf{c} = 7\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

- $\mathbf{a} \times \mathbf{b}$ **Solution:**

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -2 \\ -1 & 2 & -4 \end{bmatrix} =< -4, -10, -4 >$$

- $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$ **Solution:**

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times < 6, 5, -8 >= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -2 \\ 6 & 5 & -8 \end{bmatrix} =< -6, -36, -27 >$$

- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$

Solution:

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot < 6, 5, -8 >= 8$$

(5) Let $P(-1, 3, 1)$, $Q(0, 5, 2)$, and $R(4, 3, -1)$. Find a nonzero vector orthogonal to the plane through the points P, Q, and R.
Solution: $\vec{PQ} =< 1, 2, 1 >$ and $\vec{PR} =< 5, 0, -2 >$.

$$\vec{PQ} \times \vec{PR} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 5 & 0 & -2 \end{bmatrix} = -4\mathbf{i} + 7\mathbf{j} - 10\mathbf{k}$$

(6) Let $P(-1,3,1)$, $Q(0,5,2)$, and $R(4,3,-1)$. Find the area of the triangle PQR.
Solution: $\text{A}(PQR) = 1/2|\vec{PQ} \times \vec{PR}| = 1/2\sqrt{16+49+100} = \frac{\sqrt{165}}{2}$.

(7) Use the scalar triple product to determine whether the points $A(1,3,2)$, $B(3,-1,6)$, $C(5,2,0)$, and $D(3,6,-4)$ lie in the same plane.
Solution: $\vec{AB} =< 2,-4,4 >$, $\vec{AC} =< 4,-1,-2 >$, and $\vec{AD} =< 2,3,-6 >$. The volume is given by the triple product.

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \begin{bmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{bmatrix} = 2(12) - 4(20) + 4(14) > 0$$

Thus the points are not coplanar.

(8) Find a parametric equation for the line through $(1,-2,3)$ and $(4,5,6)$.
Solution: $\mathbf{v} =< 4-1, 5+2, 6-3 >=< 3,7,3 >$. So the line is given by $x = 1+3t$, $y = -2+7t$ and $z = 3+3t$.

(9) Let $3x - 2y + z = 1$ and $2x + y - 3z = 3$ be two planes. Find the parametric equation for the line of intersection of the planes. Also find the angle between the two planes.
Solution: First, we need to determine a point on the line of intersection. We choose to find the] point where both lines intersect the xy-plane. Setting $z = 0$ and solving for x and y, we find the point $(1,1,0)$.

Next, we need to determine the direction. For the first plane, $\mathbf{n}_1 =< 3,-2,1 >$. For the second plane, $\mathbf{n}_2 =< 2,1,-3 >$. The direction is given by

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix} =< 5,11,7 >$$

Thus the line is given by

$$(x,y,z) = (1+5t, 1+11t, 7t)$$

for $-\infty < t < \infty$.

To find the angle between the two planes, we know

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{1}{14}.$$

Solving for θ, we find the angle between the two planes to be $\arccos(1/14)$.

(10) Evaluate the limit.

$$\lim_{t\to 2}\left(\frac{t^2-2t}{t-2}\mathbf{i} + \sqrt{t+4}\mathbf{j} + \frac{\sin(\pi t)}{\ln(t-1)}\mathbf{k}\right)$$

Solution:

$$\lim_{t\to 2}\left(\frac{t^2-2t}{t-2}\mathbf{i} + \sqrt{t+4}\mathbf{j} + \frac{\sin(\pi t)}{\ln(t-1)}\mathbf{k}\right) = 2\mathbf{i} + \sqrt{6}\mathbf{j} + \pi\mathbf{k}$$

(11) Sketch the curve $\mathbf{r}(t) =< t^2, \sqrt{t}, 1 >$. Use arrows to indicate the direction in which t increases.
Solution: A picture!

(12) Find the unit tangent vector $\mathbf{T}(t)$ of $\mathbf{r}(t) =< \cos(t), -\sin(t), \sin(2t) >$ when $t = \pi/2$.
Solution:

$$\mathbf{r}'(t) =< -\sin t, -\cos t, 2\cos(2t) >$$

$$\mathbf{r}'(\pi/2) =< -1, 0, -2 >$$

$$|\mathbf{r}'(\pi/2)| = \sqrt{5}$$

$$\mathbf{T}(\pi/2) = \frac{1}{\sqrt{5}} < -1, 0, -2 >$$

(13) Find the length of the curve

$$\mathbf{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3 \right\rangle$$

for $0 \leq t \leq 1$.

Solution: We need to evaluate $L = \int_0^1 |\mathbf{r}'(t)|dt$. First,

$$\mathbf{r}'(t) = \left\langle 2, 2t, t^2 \right\rangle.$$

Hence,

$$|\mathbf{r}'(t)| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(t^2 + 2)^2} = t^2 + 2$$

Thus,

$$L = \int_0^1 |\mathbf{r}'(t)|dt = \int_0^1 (t^2 + 2)dt = \frac{t^3}{3} + 2t|_0^1 = 2\frac{1}{3}$$

(14) Find the length of the curve intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.

Solution: First, we need a parametric equation of the cylinder. To get this, we rewrite the equation of the cylinder as $x^2 + \left(\frac{y}{2}\right)^2 = 1$. From this equation it is easy to see $x = \cos t$ and $y = 2\sin t$ for $0 \leq t \leq 2\pi$. Plugging these values into the equation of the plane, we find $z = 2 - \cos t - 2\sin t$. Thus the curve is given by the vector function

$$\mathbf{r}(t) =< \cos t, 2\sin t, 2 - \cos t - 2\sin t > .$$

Hence,

$$\mathbf{r}'(t) =< -\sin t, 2\cos t, \sin t - 2\cos t > .$$

We know the length of this curve is given by

$$\begin{aligned} L &= \int_0^{2\pi} |\mathbf{r}'(t)|dt \\ &= \int_0^{2\pi} \sqrt{\sin^2 t + 4\cos^2 t + (\sin t - 2\cos t)^2 dt} \\ &= \int_0^{2\pi} \sqrt{2(\sin^2 t + 4\cos^2 t - 2\cos t \sin t)dt} \\ &= \int_0^{2\pi} \sqrt{2(4 - 2\cos t \sin t)}dt \end{aligned}$$