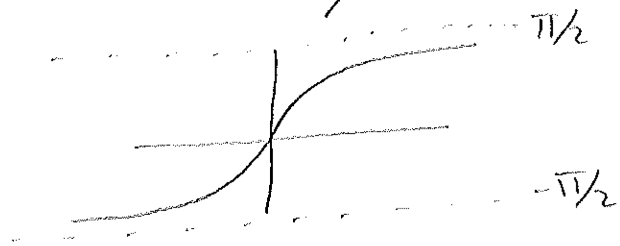
solutions to worksheet

① $\sum_{n=1}^{\infty} (\arctan n)^n$

solution: $\lim_{n \to \infty} \left((\arctan n)^n \right)^{1/n}$

$= \lim_{n \to \infty} \arctan n = \pi/2 > 1$



By root test this series diverges.

② $\sum_{n=1}^{\infty} \frac{n^n}{3^{n^2}}$

solution:

$$\lim_{n \to \infty} \left(\frac{n^n}{3^{n^2}} \right)^{1/n} = \lim_{n \to \infty} \frac{n}{3^n} \overset{L'H}{=} \lim_{n \to \infty} \frac{1}{3^n \cdot \ln 3}$$

$= 0 < 1$

By root test this series converges.

③ $\sum_{n=1}^{\infty} \frac{\ln n}{e^{n+5}}$

solution:

$$\lim_{n\to\infty} \left(\frac{\ln n}{e^{n+5}}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{(\ln n)^{1/n}}{e^{\frac{n+5}{n}}} = \frac{1}{e} < 1$$

$$\lim_{n\to\infty} (\ln n)^{1/n} = \lim_{n\to\infty} e^{\ln (\ln n)^{1/n}}$$

$$= \lim_{n\to\infty} e^{1/n \cdot \ln(\ln n)} = 1$$

$$\lim_{n\to\infty} \frac{\ln \ln n}{n} \underset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{\ln n} \cdot \frac{1}{n}}{1} = 0$$

By root test this converges!

(4) $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

solution:

The root test is inconclusive. (try it!)

Instead, this "looks like" the harmonic series.

Note $\quad n+\sqrt{n} \leq n+n = 2n.$ So

$$\frac{1}{2n} \leq \frac{1}{n+\sqrt{n}}$$

So

$$\underbrace{\sum \frac{1}{2n}}_{\text{Harmonic}} \leq \sum \frac{1}{n+\sqrt{n}}$$

As our series grows faster than a divergent series (harmonic) it must also diverge.

5) $\sum_{n=1}^{\infty} \sin e^{-n}$

solution:

$$\lim_{n\to\infty} \left(\sin e^{-n}\right)^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n} \cdot \ln \sin e^{-n}} = e^{-1} < 1$$

$$\lim_{n\to\infty} \frac{\ln \sin e^{-n}}{n} \underset{L'H}{=} \lim_{n\to\infty} \frac{1}{\sin e^{-n}} \cdot \cos e^{-n} \cdot e^{-n} \cdot -1$$

$$= \lim_{n\to\infty} \overbrace{\cos e^{-n}}^{\to 1} \cdot \frac{e^{-n}}{\sin e^{-n}} \cdot (-1)$$

$$= -1$$

$$\lim_{n\to\infty} \frac{e^{-n}}{\sin e^{-n}} \underset{L'H}{=} \lim_{n\to\infty} \frac{e^{-n} \cdot -1}{\cos e^{-n} \cdot e^{-n} \cdot -1}$$

$$= \lim_{n\to\infty} \frac{1}{\cos e^{-n}} = 1$$

$\Rightarrow$ converges by root test.