Ratio Test Handout

① $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$

solution:

$$\lim_{n\to\infty} \frac{(n+1)^{10}}{10^{n+1}} \cdot \frac{10^n}{n^{10}} = \lim_{n\to\infty} \frac{1}{10} \cdot \left(\frac{n+1}{n}\right)^{10}$$

$$= 1/10 < 1$$

By ratio test series conv.

② $\sum_{n=1}^{\infty} \frac{n^{20} \cdot 20^n}{n!}$

solution

$$\lim_{n\to\infty} \frac{(n+1)^{20} \cdot 20^{n+1}}{(n+1)!} \cdot \frac{n!}{n^{20} \cdot 20^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{20} \cdot \frac{1}{n+1} \cdot 20 = 0 < 1$$

Converges by ratio test.

③ $\sum_{n=1}^{\infty} \frac{n!}{(n+1)!}$

Solution: The ratio test is inconclusive (try it)

But $\frac{n!}{(n+1)!} = \frac{1}{n+1}$ and $\sum_{n=1}^{\infty} \frac{1}{n+1} = \frac{1}{2}+\frac{1}{3}+\cdots$

is the harmonic series so it diverges!

④ $\sum_{n=1}^{\infty} \sin(e^{-n})$

Solution:

$$\lim_{n\to\infty} \frac{\sin e^{-(n+1)}}{\sin e^{-n}} = \lim_{n\to\infty} \frac{\cos e^{-(n+1)} \cdot e^{-(n+1)} \cdot \cancel{(-1)}}{\cos e^{-n} \cdot e^{-n} \cdot \cancel{(-1)}}$$

$$= \lim_{n\to\infty} \frac{\cos e^{-(n+1)}}{\cos e^{-n}} \cdot \frac{1}{e^{-n} \cdot e^{n+1}}$$

$$= \lim_{n\to\infty} \frac{\cos e^{-(n+1)}}{\cos e^{-n}} \cdot \frac{1}{e} = \frac{1}{e} < 1$$

Conv by ratio test.

⑤ $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Solution: The ratio test is inconclusive. Note that $2n-1 < 2n$ so $\frac{1}{2n} < \frac{1}{2n-1}$

$\therefore \sum \frac{1}{2n} \le \sum \frac{1}{2n-1}$. Diverges b/c it grows faster than harmonic.

⑥ $\frac{1}{2} + \frac{2}{3^2} + \frac{4}{4^3} + \frac{8}{5^4} + \cdots = \sum_{n=0}^{\infty} \frac{2^n}{(n+2)^{n+1}}$

Solution:

$$\lim_{n\to\infty} \frac{2^{n+1}}{(n+3)^{n+2}} \cdot \frac{(n+2)^{n+1}}{2^n} = \lim_{n\to\infty} \frac{2}{(n+3)} \cdot \frac{(n+2)^{n+1}}{(n+3)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{2}{n+3} \cdot \left(\frac{n+2}{n+3}\right)^{n+1}$$

$$= 0 < 1$$

$$\lim_{n\to\infty} \left(\frac{n+2}{n+3}\right)^{n+1} = \lim_{n\to\infty} e^{(n+1)\cdot \ln\left(\frac{n+2}{n+3}\right)}$$

$$= e^{-1}$$

$$\lim_{n\to\infty} (n+1)\cdot \ln\left(\frac{n+2}{n+3}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n+2}{n+3}\right)}{1/_{n+1}}$$

$$\underset{L'H}{=} \lim_{n\to\infty} \frac{\frac{n+3}{n+2} \cdot \left(\frac{n+3-(n+2)}{(n+3)^2}\right)}{-\frac{1}{(n+1)^2}}$$

$$= \lim_{n\to\infty} -\frac{n+3}{n+2} \cdot \left(\frac{n+1}{n+3}\right)^2 = -1$$

$\Rightarrow$ Conv. by ratio test.