1) a. $f(x,y)=y^3+3x^2y-6x^2-6y^2+2$

$f_x=6xy-12x=6x(y-2)$ $\qquad$ $f_y=3y^2+3x^2-12y$

zero when $x=0$ or $y=2$

If $x=0$, $3y^2-12y=3y(y-4)=0$ so $y=0$ or 4

If $y=2$ $\quad 12+3x^2-24=3x^2-12=0$ so $x=2$ or -2

$f_{xx}=6y-12$ $\quad f_{xy}=6x$ $\quad$ Then at $(0,0)$

$f_{yy}=6y-12$

$D=(-12)^2-0=108>0$

$f_{xx}<0$ local max

at $(0,4)$

$D=(12)^2-0=108>0$

$f_{xx}>0$ local min

at $(2,2)$

$D=(0)^2-144<0$

saddle point

at $(-2,2)$

$D=(0)^2-144<0$

saddle point

b. $f(x,y)=\sin(x)\sin(y)$

Check for critical points for $-\pi\le x\le\pi,\ -\pi\le y\le\pi$

$f_x=\cos(x)\sin(y)$

zero when $x=\pi/2,-\pi/2$ or $y=0,\pi,-\pi$

$f_y=\sin(x)\cos(y)$

zero when $x=0,\pi,-\pi$ or $y=\pi/2,-\pi/2$

$f_{xx}=-\sin(x)\sin(y)=f_{yy}$

$f_{xy}=\cos(x)\cos(y)$

At $(\pi/2,\pi/2)$

$D=(-1)^2-0=1$

$f_{xx}<0$ local max

At $(\pi/2,-\pi/2)$

$D=(-1)1-0=-1$

saddle point

At $(-\pi/2,\pi/2)$

$D=1(-1)-0=-1$

saddle point

At $(-\pi/2,-\pi/2)$

$D=(1)^2-0=1$

$f_{xx}>0$ local min

At $(0,0)$, $(0,\pi)$, $(0,-\pi)$, $(\pi,0)$, (π,π), $(\pi,-\pi)$, $(-\pi,0)$, $(-\pi,\pi)$, $(-\pi,-\pi)$

$D=0-(\pm1)^2<0$, saddle point

c) $f(x,y)=e^x\cos(y)$

$f_x=e^x\cos(y)$ zero for $y=n\pi$ $\qquad$ $f_y=-e^x\sin(y)$ zero for $y=(n+\frac{1}{2})\pi$

No critical points so no local min/max

2) Note $z=2-\frac{2}{3}y-\frac{1}{3}x$. Want max for $\overset{f}{xy(2-\frac{2}{3}y-\frac{1}{3}x)}=2xy-\frac{2}{3}xy^2-\frac{1}{3}x^2y$ with $xy>0$

$f_x=2y-\frac{2}{3}y^2-\frac{2}{3}xy=2y(1-\frac{1}{3}y-\frac{1}{3}x)$ $\qquad$ $f_y=2x-\frac{4}{3}xy-\frac{1}{3}x^2=\frac{x}{3}(6-4y-x)$

zero for $y=0$ (no good) $\qquad$ zero for $x=0$ (no good)

or $1-\frac{1}{3}y-\frac{1}{3}x=0 \Rightarrow x=3-y$ $\qquad$ or $6-4y-x=0$

Combine to get

$6-4y-3+y=0 \Rightarrow y=1$ so $x=2$, $z=2-\frac{2}{3}-\frac{2}{3}=\frac{2}{3}$. Max volume is $1\cdot2\cdot\frac{2}{3}=\frac{4}{3}$

To confirm, $f_{xx}=-\frac{2}{3}y$ $\quad f_{xy}=2-\frac{4}{3}y-\frac{2}{3}x$ so $D=(-\frac{4}{3})^2-(\frac{2}{3})^2=\frac{12}{9}>0$

$f_{yy}=-\frac{4}{3}x$ $\qquad$ $f_{xx}<0$ so max

3) The distance of a point to $(4,2,0)$ is $\sqrt{(x-4)^2+(y-2)^2+z^2} = \sqrt{(x-4)^2+(y-2)^2+x^2+y^2}$

We instead minimize the distance squared $f(x,y) = (x-4)^2+(y-2)^2+x^2+y^2$

$f_x = 4x-8$ zero at $x=2$ To see is a min, $f_{xx}=4$, $f_{yy}=4$, $f_{xy}=0$ so $D=16>0$,

$f_y = 4y-4$ $\quad y=1$ $\quad f_{xx}>0$

Note $z^2=2^2+1^2=5$. Then closest points are $(2,1,\sqrt{5})$ and $(2,1,-\sqrt{5})$

4) let x,y,z be sides of the box. Surface area is $2xy+2xz+2yz=10$

so $z=\frac{10-2xy}{2x+2y}$. We want to maximize $f(x,y)=\frac{xy(10-2xy)}{2x+2y} = \frac{5xy-x^2y^2}{x+y}$

$$f_x = \frac{(x+y)(5y-2xy^2)-(5xy-x^2y^2)}{(x+y)^2}$$

$$= \frac{5xy+5y^2-2x^2y^2-2xy^3-5xy+x^2y^2}{(x+y)^2} = \frac{5y^2-x^2y^2-2xy^3}{(x+y)^2} = \frac{y^2(5-x^2-2xy)}{(x+y)^2}$$

For max, $x,y>0$ so $5-x^2-2xy=0$

$5-x^2=2xy$

Similarly, $f_y = \frac{x^2(5-y^2-2xy)}{(x+y)^2}$ so $5-y^2-2xy=0$.

Then $5-y^2-(5-x^2)=0 \Rightarrow x^2=y^2$ so $x=y$ as $x,y>0$

Then $5-3x^2=0$, so $x^2=5/3$, $x=\sqrt{5/3}=y$

Plugging in, we see $z=\frac{10-2(5/3)}{4\sqrt{5/3}} = \frac{20/3}{4\sqrt{5/3}} = \frac{5/3}{\sqrt{5/3}} = \sqrt{5/3}$

so $x=y=z=\sqrt{5/3}$, max. volume is $\frac{5}{3}\sqrt{5/3}$ (can check D if you want ☺)

5) a) We want to minimize the distance to $(0,1,0)$, which is $\sqrt{x^2+(y-1)^2+z^2}$.

Since $x=6+2y-3z$, we have distance $\sqrt{(6+2y-3z)^2+(y-1)^2+z^2}$.

To do so, we minimize the distance squared $f(y,z)=(6+2y-3z)^2+(y-1)^2+z^2$

$f_x = 4(6+2y-3z)+2(y-1) = 22+10y-12z$ when zero, have $11+5y-6z=0$

$f_y = -6(6+2y-3z)+2z = -36-12y+20z$ $\quad -18-6y+10z=0$

$-7-y+4z=0$

Then $y=4z-7$, so $11+5(4z-7)-6z=0$

$-24+14z=0$ $\quad z=12/7$, $y=-\frac{1}{7}$, $x=4/7$

b) $\vec{n} = \langle 1,-2,3\rangle$

c) $\vec{AB} = \langle 4/7, -8/7, 12/7\rangle = \frac{4}{7}\langle 1,-2,3\rangle$, which is a multiple of $\vec{n}$.

6) $f(x,y) = 4x + 6y - x^2 - y^2$ with $0 \le x \le 4,\ 0 \le y \le 5$

$f_x = 4 - 2x$ zero at $x=2$ } interior
$f_y = 6 - 2y$ $y=3$ } critical points

$f(0,y) = 6y - y^2$ inflection at $y=3$ $f(4,y) = 6y - y^2$ inflection at $y=3$

$f(x,0) = 4x - x^2$ inflection at $x=2$ $f(x,5) = -x^2 + 4x + 5$ inflection at $x=2$

Check $f(2,3) = 8 + 18 - 4 - 9 = 13$
$f(0,3) = 18 - 9 = 9$
$f(4,3) = 9$

$f(2,0) = 8 - 4 = 4$
$f(2,5) = 8 - 4 + 30 - 25 = 9$
$f(0,0) = 0$

$f(4,0) = 0$
$f(0,5) = 5$
$f(4,5) = 5$

So max at $(2,3)$, mins at $(0,0)$ and $(4,0)$

7) $f(x,y) = xe^{-x^2-y^2}$ with $x^2 + y^2 \le 4$

$f_x = e^{-x^2-y^2} - 2x^2 e^{-x^2-y^2} = e^{-x^2-y^2}(1 - 2x^2)$ zero when $x = \pm\sqrt{2}/2$

$f_y = -2xy\,e^{-x^2-y^2}$ zero when $x=0$ or $y=0$

can't happen

Interior crit. points $\left(\frac{\sqrt{2}}{2}, 0\right), \left(-\frac{\sqrt{2}}{2}, 0\right)$

Boundary is $\langle 2\cos(t), 2\sin(t) \rangle$ $0 \le t < 2\pi$

$f(2\cos(t), 2\sin(t)) = 2\cos(t)\,e^{-4\cos^2 t - 4\sin^2 t} = 2\cos(t)\,e^{-4}$ max at $t=0$ $(\cos(t)=1)$

min at $t=\pi$ $(\cos(t)=-1)$

Boundary points $(2,0), (-2,0)$

$f\left(\frac{\sqrt{2}}{2}, 0\right) = \frac{\sqrt{2}}{2} e^{-\frac{1}{2}}$ $f(2,0) = 2e^{-4}$

$f\left(-\frac{\sqrt{2}}{2}, 0\right) = -\frac{\sqrt{2}}{2} e^{-1/2}$ $f(-2,0) = -2e^{-4}$

so max $= \frac{\sqrt{2}}{2} e^{-1/2} > 2e^{-4}$

min $= -\frac{\sqrt{2}}{2} e^{-1/2} < -2e^{-4}$