1. [8 points] Determine if the following sequence converges or diverges. If it converges, find the limit.

$$\left\{ \frac{\ln(n^3+n)}{n^2+1} \right\}_{n=1}^{\infty}$$

Note $\lim_{n\to\infty} \ln(n^3+n) \to \infty$

$\lim_{n\to\infty} n^2+1 \to \infty$

To compare, we use L'Hopital's rule

$$\lim_{n\to\infty} \frac{\ln(n^3+n)}{n^2+1} = \lim_{n\to\infty} \frac{3n^2+1/n^3+n}{2n}$$

$$= \lim_{n\to\infty} \frac{3n^2+1}{2n^4+2n^2}$$

$$= \lim_{n\to\infty} \frac{3n^2/n^4 + 1/n^4}{2n^4/n^4 + 2n^2/n^4}$$

$$= \lim_{n\to\infty} \frac{3/n^2 + 1/n^4}{2+2/n^2} = 0$$

2. Find the sum of the following series:

(a) [7 points] $\sum_{n=0}^{\infty} \frac{2^{n+3}}{3^{2n}}$

$$\sum_{n=0}^{\infty} \frac{2^{n+3}}{3^{2n}} = \sum_{n=0}^{\infty} \frac{2^3 2^n}{(3^2)^n}$$

$$= 8 \sum_{n=0}^{\infty} \frac{2^n}{9^n} = 8 \sum_{n=0}^{\infty} \left(\frac{2}{9}\right)^n$$

$$= \frac{8}{1 - \frac{2}{9}}$$

$$= \frac{72}{7}$$

(b) [7 points] $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

Recall $\ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$

so

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \ln(n) - \ln(n+1)$$

$$= \lim_{k\to\infty} \sum_{n=1}^{k} \ln(n) - \ln(n+1)$$

telescoping $\overset{\uparrow}{=} \lim_{k\to\infty} \ln(1) - \ln(k)$

$$= \lim_{k\to\infty} -\ln(k) \quad \text{diverges to } -\infty$$

3. [7 points] Determine if the following series converges or diverges. Explain your reasoning.

$$\sum_{n=0}^{\infty} \frac{2n}{n^2-5}$$

Note $\frac{2n}{n^2-5} > \frac{2n}{n^2} = \frac{2}{n}$ for $n > 2$

Since $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges to infinity, (it is harmonic)

by the comparison test we see

$\sum_{n=0}^{\infty} \frac{2n}{n^2-5}$ diverges

4. [7 points] Determine if the following series converges or diverges. Explain your reasoning.

$$\sum_{n=1}^{\infty}\left(\cos\left(\frac{1}{n}\right)\right)^n$$

Note: $\left(\cos\left(\frac{1}{n}\right)\right)^n = e^{\ln\left(\left(\cos\left(\frac{1}{n}\right)\right)^n\right)}$

$$= e^{n\ln(\cos(1/n))}$$

So $\lim_{n\to\infty}\left(\cos\left(\frac{1}{n}\right)\right)^n = \lim_{n\to\infty} e^{n\ln(\cos(1/n))}$

$$= e^{\lim_{n\to\infty} n\ln(\cos(1/n))}$$

We compute $\lim_{n\to\infty} n\ln(\cos(1/n)) = \lim_{n\to\infty}\frac{\ln(\cos(1/n))}{1/n}$

L'Hopital's Rule $\overset{\curvearrowright}{=} \lim_{n\to\infty}\frac{\frac{1}{\cos(1/n)}\left(-\sin(1/n)\right)\left(-\frac{1}{n^2}\right)}{-1/n^2}$

$$= \lim_{n\to\infty}\frac{-\sin(1/n)}{\cos(1/n)} = \frac{0}{1} = 0,$$

So $\lim_{n\to\infty}(\cos(1/n))^n = e^0 = 1.$

By the test for divergence, the series must diverge

5. [7 points] Determine if the following series converges or diverges. Explain your reasoning.

$$\frac{\ln 2}{3} + \frac{\ln 3}{3^2} + \frac{\ln 4}{3^3} + \frac{\ln 5}{3^4} + \cdots = \sum_{n=1}^{\infty} \frac{\ln(n+1)}{3^n}$$

Using the ratio test, we compute

$$\lim_{n\to\infty} \left| \frac{\ln(n+2)}{3^{n+1}} \Big/ \frac{\ln(n+1)}{3^n} \right| = \lim_{n\to\infty} \left| \frac{3^n}{3^{n+1}} \frac{\ln(n+2)}{\ln(n+1)} \right|$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{\ln(n+2)}{\ln(n+1)} = 1/3,$$

so the series converges by the ratio test.

6. [10 points] Compute the integral

$$\int \ln(2x+1)dx.$$

Let $u = 2x+1$, $du = 2dx$, so $dx = \frac{du}{2}$

Then

$$\int \ln(2x+1)dx = \int \ln(u)\frac{du}{2}$$

$$= \frac{1}{2}\int \ln(u)du$$

Let $f(u) = \ln(u)$, $g(u) = u$

$f'(u) = \frac{1}{u}du$ $g'(u) = du$

Then $\frac{1}{2}\left(\underbrace{\int \ln(u)du}_{f \cdot g'}\right) = \frac{1}{2}\left(\underbrace{u \ln(u)}_{g \cdot f} - \underbrace{\int u \cdot \frac{1}{u}du}_{g \cdot f'}\right)$

$$= \frac{1}{2}u\ln(u) - \frac{u}{2} + C$$

Since $u = 2x+1$, we have

$$\int \ln(2x+1)dx = \frac{(2x+1)}{2}\ln(2x+1) - \frac{2x+1}{2} + C$$

$$= x\ln(2x+1) + \frac{1}{2}\ln(2x+1) - x + C$$

7. The Taylor series for $\ln(x+2)$ centered at $a=3$ is

$$\ln 5 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot 5^n}(x-3)^n.$$

(a) [10 points] Find the interval of convergence.

$\boxed{(-2, 8]}$

Use the ratio test:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-3)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n\, 5^n}{(-1)^{n+1}(x-3)^n} \right| = \lim_{n\to\infty} \left| \frac{-(x-3)}{5} \frac{n}{n+1} \right| = \frac{|x-3|}{5}$$

For $-1 < \frac{x-3}{5} < 1$, equivalently $-5 < x-3 < 5$

equivalently $-2 < x < 8$,

the series converges. For $\frac{x-3}{5} < -1$ $(x < -2)$ or $\frac{x-3}{5} > 1$ $(x > 8)$,

the series diverges. For $x=-2$, have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{(-5)^n}{5^n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

For $x=8$, have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{5^n}{5^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

As $\lim_{n\to\infty} \frac{1}{n} = 0$, $\frac{1}{n} > 0$ the series converges by the alternating series test.

(b) [4 points] Can you use this series to approximate $\ln(11)$? Explain your reasoning.

No. Since 11 is not in the interval of convergence, this series cannot be used to approximate $\ln(11)$ directly.

8. [15 points] Find the Taylor series for $\cos(2x)$ around $a = \pi/4$. Express your final answer in sigma notation.

Want $\cos(2x) = \sum_{n=0}^{\infty} C_n (x - \pi/4)^n$

w/ $C_n = \dfrac{f^{(n)}(\pi/4)}{n!}$

Let

		at $\pi/4$
$f(x) = \cos(2x)$		0
$f'(x) = -2\sin(2x)$		-2
$f''(x) = -4\cos(2x)$		0
$f^{(3)}(x) = 8\sin(2x)$		8
$f^{(4)}(x) = 16\cos(2x)$		0
$\vdots$		$\vdots$

so $C_n = 0$ n even

and

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{2^{2n+1}(-1)^{n+1}}{(2n+1)!}(x - \pi/4)^{2n+1}$$

9. [10 points] Evaluate the following limit using Taylor series. Other methods will receive no credit.

$$\lim_{x\to 0} \frac{6\sin(2x) - 12x + 8x^3}{x^5}$$

Recall $\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$

The first terms are $2x,\ \frac{4}{3}x^3,\ \frac{2^5}{5!}x^5$

so $6\sin(2x) - 12x + 8x^3 = 12x - 8x^3 + \frac{8}{5}x^5 + \sum_{\underline{\underline{n=3}}}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$
$-12x + 8x^3$

$= \frac{8}{5}x^5 + \sum_{n=3}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$

so $\lim_{x\to 0} \frac{6\sin(2x) - 12x + 8x^3}{x^5}$

$= \lim_{x\to 0} \frac{\frac{8}{5}x^5 + \sum_{\underline{\underline{n=3}}}^{\infty} \frac{(-1)^n 2^{(2n+1)} x^{2n+1}}{(2n+1)!}}{x^5}$

≥ 2 for $n \geq 3$

$= \lim_{x\to 0} \frac{8}{5} + \sum_{n=3}^{\infty} \frac{(-1) 2^{2n+1} x^{2n-4}}{(2n+1)!} = \boxed{\frac{8}{5}}$

10. [8 points] If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent series is it true that

$$\sum_{n=1}^{\infty} a_n b_n$$

is also a convergent series? If it is convergent explain why, if not give a counterexample.

let $a_n = \frac{(-1)^n}{\sqrt{n}} = b_n$

Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

converge by the AST as $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$,

alternating series test

but $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{\sqrt{n}\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic)

If you assume $a_n, b_n > 0$ (as some did),

then eventually for n large $a_n < 1$

so $a_n b_n < b_n$.

The series would then converge by the comparison test.