1. (13 points) Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{(x-3)^n}{5^n n^5}$.

Ratio test:

$$\lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{5^{n+1}(n+1)^5} \cdot \frac{5^n n^5}{(x-3)^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x-3|\, n^5}{5\,(n+1)^5} = \frac{|x-3|}{5}$$

The series converges when $\frac{|x-3|}{5} < 1$

$$\Rightarrow \quad |x-3| < 5$$

$$-5 < x-3 < 5$$

$$-2 < x < 8$$

Endpoints:

$x = 8$: $\sum_{n=1}^{\infty} \frac{(8-3)^n}{5^n n^5} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ p-series with $p = 5 > 1$ converges

$x = -2$: $\sum_{n=1}^{\infty} \frac{(-2-3)^n}{5^n n^5} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$

Alternating series test: $\frac{1}{n^5}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n^5} = 0$ so the series converges

$$\boxed{\text{Interval of Convergence: } -2 \le x \le 8}$$

2. (6 points each) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent. State any tests you use.

(a) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{3^{n+1}\sqrt{n}}$ $= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^{n-1}}{3^{n+1} \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{9\sqrt{n}}$

Alternating Series Test: $a_n = \frac{1}{9\sqrt{n}}$ is decreasing and $\lim_{n\to\infty} \frac{1}{9\sqrt{n}} = 0$ so the series converges.

But $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{9\sqrt{n}} \right| = \frac{1}{9} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ and

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series $(p = \frac{1}{2} < 1)$

So the series is $\boxed{\text{conditionally convergent.}}$

(b) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n+1}{3n}\right)^{3n}$

Root test:

$$\lim_{n\to\infty} \sqrt[n]{\left| (-1)^n \left(\frac{n+1}{3n}\right)^{3n} \right|} = \lim_{n\to\infty} \left(\frac{n+1}{3n}\right)^3$$

$$= \left(\frac{1}{3}\right)^3 = \frac{1}{27} < 1$$

So the series is $\boxed{\text{absolutely convergent}}$

3. (12 points) Find the Maclaurin series for $f(x) = \dfrac{x^3}{(2+4x)^2}$, and give its radius of convergence. The first four nonzero terms of the series are sufficient.

$$f(x) = x^3 \cdot \frac{1}{(2+4x)^2}$$

$$|-2x| \leq 1$$

$$|x| \leq \frac{1}{2}$$

$$R = \frac{1}{2}$$

$$\int \frac{1}{(2+4x)^2}\,dx = -\frac{1}{4}\cdot\frac{1}{2+4x}$$

$$-\frac{1}{4}\cdot\frac{1}{2+4x} = -\frac{1}{8}\cdot\frac{1}{1-(-2x)} = -\frac{1}{8}\sum_{n=0}^{\infty}(-2x)^n$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}2^{n-3}x^n$$

$$\frac{1}{(2+4x)^2} = \frac{d}{dx}\left(-\frac{1}{4}\cdot\frac{1}{2+4x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^{n+1}2^{n-3}x^n\right)$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}2^{n-3}nx^{n-1}$$

$$f(x) = \frac{x^3}{(2+4x)^2} = x^3\sum_{n=1}^{\infty}(-1)^{n+1}2^{n-3}nx^{n-1}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}2^{n-3}nx^{n+2}$$

$$= \frac{x^3}{4} - x^4 + 3x^5 - 8x^6 + \cdots$$

The radius of convergence is $R = \frac{1}{2}$ Since taking the derivative does not change it.

4. (10 points) Find the first four nonzero terms of the Taylor series for $f(x) = \ln x$ centered at $a = 2$.

Use $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$

$f(x) = \ln x \qquad f(2) = \ln 2$

$f'(x) = \frac{1}{x} \qquad f'(2) = \frac{1}{2}$

$f''(x) = -\frac{1}{x^2} \qquad f''(2) = -\frac{1}{4}$

$f'''(x) = \frac{2}{x^3} \qquad f'''(2) = \frac{2}{8} = \frac{1}{4}$

The Taylor series is

$$\ln 2 + \frac{1}{2}(x-2) - \frac{1}{4\cdot 2!}(x-2)^2 + \frac{1}{4\cdot 3!}(x-2)^3 + \cdots$$

$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 + \cdots$$

5. (6 points each) Consider the points $P(2,0,-3)$, $Q(3,1,0)$ and $R(5,2,2)$.

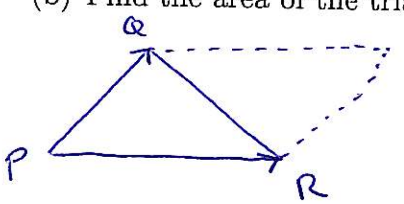
(a) Find a vector orthogonal to the plane through P, Q and R.

$$\vec{PQ} = \langle 1, 1, 3 \rangle \qquad \vec{PR} = \langle 3, 2, 5 \rangle$$

The plane through P, Q and R contains $\vec{PQ}$ and $\vec{PR}$. The cross product $\vec{PQ} \times \vec{PR}$ gives a vector orthogonal to the plane.

$$\langle 1, 1, 3 \rangle \times \langle 3, 2, 5 \rangle = \boxed{\langle -1, 4, -1 \rangle}$$

(b) Find the area of the triangle PQR.



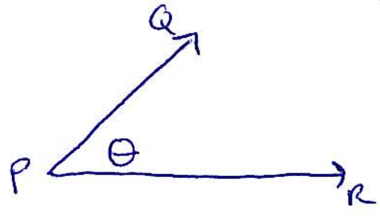
The area of the triangle PQR is half of the area of the parallelogram determined by $\vec{PQ}$ and $\vec{PR}$, so

$$\text{Area} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} |\langle -1, 4, -1 \rangle|$$

$$= \frac{1}{2}\sqrt{1+16+1}$$

$$= \boxed{\frac{1}{2}\sqrt{18}}$$

(c) Find the cosine of the angle between vectors PQ and PR.



$$\cos\theta = \frac{\vec{PQ} \cdot \vec{PR}}{|\vec{PQ}|\,|\vec{PR}|}$$

$$\cos\theta = \frac{\langle 1,1,3 \rangle \cdot \langle 3,2,5 \rangle}{\sqrt{1+1+9}\ \sqrt{9+4+25}} = \frac{3+2+15}{\sqrt{11}\ \sqrt{38}}$$

$$= \boxed{\frac{20}{\sqrt{418}}}$$

6. Consider the planes $P1 : x + y - z = 0$ and $P2 : x - 3y + z = 2$.

(a) (12 points) Find parametric equations for the line of intersection of $P1$ and $P2$.

The vector of the line is contained in both planes, so it is orthogonal to both normal vectors.

So $\bar{v} = \langle 1, 1, -1 \rangle \times \langle 1, -3, 1 \rangle = \langle -2, -2, -4 \rangle$

To find a point on the line, set $y = 0$:

$$\begin{array}{l} x - z = 0 \\ x + z = 2 \end{array} \Rightarrow \begin{array}{l} 2x = 2 \\ x = 1,\ z = 1 \\ (1, 0, 1) \end{array}$$

$\boxed{\text{Line: } x = -2t + 1,\ y = -2t,\ z = -4t + 1}$

* There are other possible answers depending on your choice of point and vector.

(b) (8 points) Find an equation for the plane containing the origin that is orthogonal to both $P1$ and $P2$.

If a plane is orthogonal to both P1 and P2, then its normal vector is orthogonal to the normal vectors of both P1 and P2.

So the normal vector of the plane is given by $\langle 1, 1, -1 \rangle \times \langle 1, -3, 1 \rangle = \langle -2, -2, -4 \rangle$

Then an equation for the plane through $(0,0,0)$ with this normal vector is

$-2x - 2y - 4z = 0$, or $x + y + 2z = 0$.

7. (3 points each) SHORT ANSWER: For each of the following, you do not need to justify your answer, and no partial credit will be given.

(a) Find the interval of convergence of the Taylor series $\sum_{n=1}^{\infty}(-1)^n n!(x-5)^n$.

Use Ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)!(x-5)^{n+1}}{(-1)^n n!(x-5)^n}\right| = \lim_{n\to\infty} n|x-5|$$

the limit is ∞ unless $x=5$

$\boxed{\text{Interval of convergence is } \{5\}}$

(b) Find the sum of the series $1+2+\frac{4}{2!}+\frac{8}{3!}+\frac{16}{4!}+\cdots$.

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$1+2+\frac{4}{2!}+\frac{8}{3!}+\frac{16}{4!}+\cdots = \sum_{n=0}^{\infty}\frac{2^n}{n!} = \boxed{e^2}$$

(c) If **a** and **b** are vectors with $|\mathbf{a}|=3$, $|\mathbf{b}|=4$ and $|\mathbf{a}\times\mathbf{b}|=12$, then what is $\mathbf{a}\cdot\mathbf{b}$?

$|\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}|\sin\theta$, so

$12 = 3\cdot 4\cdot \sin\theta \implies \sin\theta = 1$

so $\theta = \frac{\pi}{2}$.

Then $\bar{a}$ and $\bar{b}$ are orthogonal, so

$\boxed{\bar{a}\cdot\bar{b} = \bar{0}}$

(d) If $f(x) = \sum_{n=0}^{\infty} \frac{(x-2)^{6n+2}}{4^n}$, find $f^{(44)}(2)$. (You don't need to simplify your answer.)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

When $n=7$, $6n+2 = 44$, so

$$\frac{1}{4^7} = \frac{f^{(44)}(2)}{44!}.$$ Then

$$\boxed{f^{(44)}(2) = \frac{44!}{4^7}}$$

(e) Give an inequality representing the spherical shell centered at $(5, 2, 1)$ with inner radius 1 and outer radius 4.

$$1 \leq (x-5)^2 + (y-2)^2 + (z-1)^2 \leq 16$$