1. (12 points) Evaluate the integral $\int x^2 e^{-x}\, dx$.

Integration by Parts:

$u = x^2 \qquad dv = e^{-x}\, dx$

$du = 2x\, dx \qquad v = -e^{-x}$

$$\int x^2 e^{-x}\, dx = -x^2 e^{-x} + 2\int x e^{-x} \cdot dx$$

$u = x \qquad dv = e^{-x}\, dx$

$du = dx \qquad v = -e^{-x}$

$$\int x^2 e^{-x}\, dx = -x^2 e^{-x} + 2\left[-x e^{-x} + \int e^{-x}\, dx\right]$$

$$= \boxed{-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C}$$

2. (10 points) Evaluate the integral $\int \sin^5(3x)\,dx$.

Use $\sin^2 x + \cos^2 x = 1$:

$$\int \sin^5(3x)\,dx = \int (1-\cos^2(3x))^2 \sin(3x)\,dx$$

Substitute $u = \cos(3x)$

$du = -3\sin(3x)\,dx$

$$\int \sin^5(3x)\,dx = \int (1-u^2)^2 \left(-\frac{1}{3}\right) du$$

$$= -\frac{1}{3}\int u^4 - 2u^2 + 1\,du$$

$$= -\frac{1}{3}\left(\frac{u^5}{5} - \frac{2u^3}{3} + u\right) + C$$

$$= \boxed{-\frac{1}{3}\left(\frac{\cos^5(3x)}{5} - \frac{2\cos^3(3x)}{3} + \cos(3x)\right) + C}$$

3. (12 points) Evaluate the integral $\int \frac{x^3}{\sqrt{x^2+4}}\,dx$.

Trig. substitution: $x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int \frac{8\tan^3\theta}{\sqrt{4\tan^2\theta+4}} \cdot 2\sec^2\theta\, d\theta$$

$$= \int \frac{8\tan^3\theta}{2\sec\theta} \cdot 2\sec^2\theta\, d\theta$$

$$= 8\int \tan^3\theta \sec\theta\, d\theta$$

$$= 8\int (\sec^2\theta - 1)\tan\theta\sec\theta\, d\theta$$

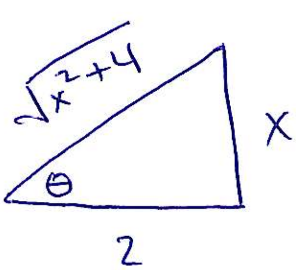
$u = \sec\theta$

$\rightarrow$ $$= 8\int u^2 - 1\, du$$

$du = \sec\theta\tan\theta\, d\theta$

$$= 8\left(\frac{u^3}{3} - u\right) + C$$

$$= 8\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C$$



$$= 8\left(\frac{1}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - \frac{\sqrt{x^2+4}}{2}\right) + C$$

$$= \boxed{\frac{(x^2+4)^{3/2}}{3} - 4\sqrt{x^2+4} + C}$$

4. (12 points) Evaluate the integral $\int_0^2 x^2 \ln x \, dx$.

The function $f(x) = x^2 \ln x$ is undefined at $x = 0$, so the integral is improper.

$$\int_0^2 x^2 \ln x \, dx = \lim_{t \to 0^+} \int_t^2 x^2 \ln x \, dx$$

Use integration by parts: $u = \ln x \quad dv = x^2 \, dx$

$du = \frac{1}{x} dx \quad v = \frac{x^3}{3}$

$$\int_0^2 x^2 \ln x \, dx = \lim_{t \to 0^+} \left(\frac{x^3}{3} \ln x \Big|_t^2 - \int_t^2 \frac{x^2}{3} dx \right)$$

$$= \lim_{t \to 0^+} \left(\frac{x^3}{3} \ln x - \frac{x^3}{9} \Big|_t^2 \right)$$

$$= \lim_{t \to 0^+} \left[\left(\frac{8}{3} \cdot \ln 2 - \frac{8}{9} \right) - \left(\frac{t^3}{9} \cdot \ln t - \frac{t^3}{9} \right) \right]$$

$$= \frac{8 \ln 2}{3} - \frac{8}{9} - \lim_{t \to 0^+} \frac{t^3}{9} \cdot \ln t + \lim_{t \to 0^+} \frac{t^3}{9}$$

$\lim_{t \to 0^+} \frac{t^3}{9} = 0$. Using l'Hopital's Rule (since $t^3 \to 0$ and $\ln t \to -\infty$):

$$\lim_{t \to 0^+} \frac{t^3}{9} \cdot \ln t = \lim_{t \to 0^+} \frac{\ln t}{9t^{-3}} = \lim_{t \to 0^+} \frac{1/t}{-27t^{-4}} = \lim_{t \to 0^+} -\frac{t^3}{27} = 0$$

So $$\boxed{\int_0^2 x^2 \ln x \, dx = \frac{8 \ln 2}{3} - \frac{8}{9}}$$

5. (6 points each) Determine if the series (a)–(d) converge or diverge. Clearly state any tests you use. A correct conclusion with incorrect reasoning will be considered wrong.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}$

Alternating Series Test:

$\lim_{n \to \infty} \frac{1}{n^n} = 0$, and

$\frac{1}{n^n} > \frac{1}{(n+1)^{n+1}}$ for all n, so

the sequence $\frac{1}{n^n}$ is decreasing.

So $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}$ converges.

(b) $\sum_{n=2}^{\infty} \frac{n^3+5}{n^4+2n^2+1}$

Limit Comparison Test:

All the terms in the series are positive.

Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, the harmonic series, which diverges.

$$\lim_{n \to \infty} \left(\frac{n^3+5}{n^4+2n^2+1} \cdot n \right) = \lim_{n \to \infty} \frac{n^4+5n}{n^4+2n^2+1} = 1$$

Since $0 < 1 < \infty$, we conclude that both series diverge, so

$\sum_{n=2}^{\infty} \frac{n^3+5}{n^4+2n^2+1}$ diverges.

(c) $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ Integral Test:

The function $f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}}$ is continuous and positive on $[1,\infty)$. It is also decreasing since

$$\frac{e^{-\sqrt{x}}}{\sqrt{x}} = \frac{1}{\sqrt{x}\, e^{\sqrt{x}}},$$ which decreases as x increases.

$$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx = \lim_{t\to\infty} \int_1^{t} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx = \lim_{t\to\infty} \int_1^{\sqrt{t}} 2e^{-u}\, du$$

$$= \lim_{t\to\infty} -2e^{-u}\Big|_1^{\sqrt{t}} = \lim_{t\to\infty} \left(-2e^{-\sqrt{t}} + \frac{2}{e}\right) = \frac{2}{e}$$

Since the integral converges, the series also converges.

(d) $\sum_{n=1}^{\infty} \frac{2+\sin\left(\frac{n\pi}{2}\right)}{3^n}$ Comparison Test:

$-1 \le \sin\left(\frac{n\pi}{2}\right) \le 1$ so the series has positive terms, and

$$\frac{2+\sin\left(\frac{n\pi}{2}\right)}{3^n} \le \frac{3}{3^n} = \frac{1}{3^{n-1}}$$

$\sum_{n=1}^{\infty} \frac{1}{3^{n-1}}$ is a convergent geometric series since $r = \frac{1}{3} < 1$.

Therefore $\sum_{n=1}^{\infty} \frac{2+\sin\left(\frac{n\pi}{2}\right)}{3^n}$ converges.

6. (6 points each) Determine if each series below converges or diverges. If it converges, find its sum.

(a) $\sum_{n=2}^{\infty} \frac{3^{n-1}}{5^{n+1}}$ $= \sum_{n=2}^{\infty} \frac{1}{25}\left(\frac{3}{5}\right)^{n-1}$ This is a geometric series with $r = \frac{3}{5} < 1$ so it converges.

$$\sum_{n=2}^{\infty} \frac{3^{n-1}}{5^{n+1}} = \frac{3}{5^3} + \frac{3^2}{5^4} + \frac{3^3}{5^5} + \cdots$$

so $a = \frac{3}{5^3} = \frac{3}{125}$. The sum is $\frac{a}{1-r} = \frac{3/125}{1-3/5} = \boxed{\frac{3}{50}}$

(b) $\sum_{n=1}^{\infty}(-1)^n\pi$ Test for Divergence:

$\lim_{n\to\infty} (-1)^n \pi$ does not exist, so the series diverges.

(c) $\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{(n+2)^2}\right)$ Telescoping series

$$S_n = \left(1 - \cancel{\frac{1}{3^2}}\right) + \left(\frac{1}{2^2} - \cancel{\frac{1}{4^2}}\right) + \left(\cancel{\frac{1}{3^2}} - \cancel{\frac{1}{5^2}}\right) + \cdots + \left(\cancel{\frac{1}{(n-1)^2}} - \frac{1}{(n+1)^2}\right) + \left(\cancel{\frac{1}{n^2}} - \frac{1}{(n+2)^2}\right)$$

$$S_n = 1 + \frac{1}{4} - \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2}$$

$\lim_{n\to\infty} S_n = 1 + \frac{1}{4} = \frac{5}{4}$. Since $\lim_{n\to\infty} S_n$ exists and is finite, the series converges and the sum is $5/4$.

7. (3 points each) SHORT ANSWER: For each of the following, you do not need to justify your answer, and no partial credit will be given.

(a) Find the limit of the sequence $a_n = \dfrac{3^n}{n!}$.

$$\frac{3^n}{n!} = \frac{3 \cdot 3 \cdot \boxed{3 \cdot \cdots \cdot 3} \cdot 3}{1 \cdot 2 \cdot \boxed{3 \cdot \cdots \cdot (n-1)} \cdot n} < \frac{3 \cdot 3 \cdot 3}{1 \cdot 2 \cdot n} = \frac{27}{2n}$$

Since $\longrightarrow$ is less than 1. Then

$0 \le \dfrac{3^n}{n!} \le \dfrac{27}{2n}$, and $\lim_{n\to\infty} \dfrac{27}{2n} = 0$, so by the squeeze Theorem, $\lim_{n\to\infty} a_n = 0$.

(b) What substitution should be used to evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{25 - 9x^2}}$?

Trig. substitution: $x = \frac{5}{3} \sin \theta$

(c) For which values of p does the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

For all $p > 1$

(Recall this is how we determined which p-series converge.)

(d) How many terms of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2}$ do we need to add in order to estimate the sum of the series with error less than 0.01?

The series should have been $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2}$, so

the terms are: $-\frac{1}{4} + \frac{1}{16} - \frac{1}{36} + \frac{1}{64} - \frac{1}{100} + \frac{1}{144} - \cdots$

Then $\frac{1}{144} < 0.01$ so you need to use

S_5 (5 terms) in your estimate.