

## 1 Linear Equations and Matrices

Many problems in mathematics eventually lead to solving a system of linear equations. We have already run into some of them. We are about to learn a method of solving systems of linear equations using matrices.

Matrices are very useful in mathematics and in applied mathematics, and not just for solving systems of linear equations. For one thing, in multivariable calculus, the derivative of a function at a point is not just a single number, it is a matrix. The operations with matrices we will learn here, in particular matrix multiplication, are also widely applicable.

## 2 Operations on Systems of Linear Equations

### 2.1 An example

**Example:** Find the intersection of the three planes with equations

$$2x - 4y + 3z = 1$$

$$x + y + z = 3$$

$$3x - y - z = 1$$

**Solution:** Generally we expect the intersection of three planes to be a single point, but this does not always happen; three planes can have no intersection, or intersect in a line, or even in a plane (if the three planes are all the same.) So our problem is to find all the points  $(x, y, z)$  in the intersection of these planes. This is the same as finding all values of  $x$ ,  $y$  and  $z$  for which all three equations hold. We call this **solving a system of linear equations**. A value of  $(x, y, z)$  for which all three equations hold is a **solution** of the system; solving the system means finding all the solutions.

Our method is going to be to convert this system into a simpler but **equivalent** system. Two systems of linear equations are equivalent if they have exactly the same solutions. For example, obviously we don't change anything about the solutions to the system if we multiply an equation through

by a constant (as long as we don't multiply by zero.) So we'll multiply the first equation by  $\frac{1}{2}$ . Our new system, equivalent to the old one, is:

$$\begin{aligned}x - 2y + \left(\frac{3}{2}\right)z &= \frac{1}{2} \\x + y + z &= 3 \\3x - y - z &= 1\end{aligned}$$

The next thing we'll do is subtract equation 1 from equation 2 (so our new equation 2 is the old equation 2 minus the old equation 1, and equations 1 and 3 are unchanged.) This also does not change anything about the solutions to our system of equations; the new system is still equivalent to the original one:

$$\begin{aligned}x - 2y + \left(\frac{3}{2}\right)z &= \frac{1}{2} \\3y - \left(\frac{1}{2}\right)z &= \frac{5}{2} \\3x - y - z &= 1\end{aligned}$$

We'll subtract 3 times equation 1 (or add  $-3$  times equation 1) to equation 3, to eliminate  $x$  from equation 3:

$$\begin{aligned}x - 2y + \left(\frac{3}{2}\right)z &= \frac{1}{2} \\3y - \left(\frac{1}{2}\right)z &= \frac{5}{2} \\5y - \left(\frac{11}{2}\right)z &= -\frac{1}{2}\end{aligned}$$

Now we have an  $x$  term, with coefficient 1, in equation 1, and we have eliminated  $x$  from the other two equations. The next step is to try for a

$y$  term, with coefficient 1, in equation 2, and eliminate  $y$  from the other equations. We'll use the same two kinds of operations: Multiply an equation by a non-zero constant. Add a constant multiple of one equation to another equation. See if you can follow the steps:

$$\begin{aligned}x - 2y + \left(\frac{3}{2}\right)z &= \frac{1}{2} \\y - \left(\frac{1}{6}\right)z &= \frac{5}{6} \\5y - \left(\frac{11}{2}\right)z &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}x + \left(\frac{7}{6}\right)z &= \frac{13}{6} \\y - \left(\frac{1}{6}\right)z &= \frac{5}{6} \\5y - \left(\frac{11}{2}\right)z &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}x + \left(\frac{7}{6}\right)z &= \frac{13}{6} \\y - \left(\frac{1}{6}\right)z &= \frac{5}{6} \\-\left(\frac{28}{6}\right)z &= -\frac{28}{6}\end{aligned}$$

First we multiplied the second equation by  $\frac{1}{3}$ . Then we added 2 times the second equation to the first equation. Then we added  $-5$  times the second equation to the third equation.

Now we'll take care of  $z$ :

$$\begin{aligned}x + \left(\frac{7}{6}\right)z &= \frac{13}{6} \\y - \left(\frac{1}{6}\right)z &= \frac{5}{6} \\z &= 1\end{aligned}$$

$$\begin{aligned}x &= 1 \\y - \left(\frac{1}{6}\right)z &= \frac{5}{6} \\z &= 1\end{aligned}$$

$$\begin{aligned}x &= 1 \\y &= 1 \\z &= 1\end{aligned}$$

The three planes intersect in the single point  $(1, 1, 1)$ .

## 2.2 The general method

Whenever we have a system of simultaneous linear equations to solve we try to do something like this. We start with a system:

$$\begin{aligned}a_1x_1 + a_2x_2 + \cdots + a_nx_n &= a_0 \\b_1x_1 + b_2x_2 + \cdots + b_nx_n &= b_0 \\c_1x_1 + c_2x_2 + \cdots + c_nx_n &= c_0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot\end{aligned}$$

We convert it to an equivalent system by a sequence of operations. There are three allowable operations:

1. Multiply an equation by a non-zero constant.
2. Add a multiple of one equation to another equation.
3. Change the order in which the equations are written by interchanging two equations.

We are hoping, in the ideal situation, to end up with an equivalent system that looks like this:

$$x_1 = \textit{number}_1$$

$$x_2 = \textit{number}_2$$

$$x_3 = \textit{number}_3$$

.

.

.

In other words, we are hoping to find a unique solution to our system. Of course this doesn't always happen; there isn't always a unique solution. If our system consists of the equations of three planes that intersect in a line, there are infinitely many solutions (all the points on the line.) We'll see an example of that in a moment. First, a matrix method of organizing things, that saves writing (if nothing else) and lets us make things very systematic.

## 3 A Matrix Method for Solving Systems of Linear Equations

### 3.1 From System of Equations to Matrix

In solving our system of linear equations

$$\begin{aligned}2x - 4y + 3z &= 1 \\x + y + z &= 3 \\3x - y - z &= 1\end{aligned}$$

we ended up writing many equivalent versions of this system as we went through the various steps. Every version looked something like

$$\begin{aligned}\_x + \_y + \_z &= \_ \\ \_x + \_y + \_z &= \_ \\ \_x + \_y + \_z &= \_ \end{aligned}$$

with different coefficients and constant terms put into the blank spaces. We could choose not to write all the extra stuff, and simply work with those coefficients and constant terms. In order to keep track of which coefficient or constant goes where, we arrange them into a rectangular array called a **matrix**.

The “coefficient matrix” of a system is the matrix whose rows are the coefficients of  $x$ ,  $y$  and  $z$  in the different equations, arranged in the same order in each row; in this example it is natural always to put the coefficient of  $x$  first, then the coefficient of  $y$ , then the coefficient of  $z$ . So we write the coefficient matrix like this:

$$A = \begin{pmatrix} 2 & -4 & 3 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{pmatrix}.$$

This is a  $3 \times 3$  matrix, meaning that it has three rows (one for each equation) and three columns (one for each variable.) The entry in row  $i$ , column  $j$  of  $A$  (sometimes denoted  $a_{ij}$ ) comes from equation number  $i$ , and is the coefficient

of variable number  $j$  in that equation. So the entry in row 3, column 1 is 3 because in equation 3 the 1<sup>st</sup> variable  $x$  has a coefficient of 3.

The constant terms from all the equations can be arranged into matrix consisting of a single column. Such a matrix is also called a column vector. In this case we have a  $3 \times 1$  matrix, three rows and one column.

$$B = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

Finally, we obtain the *augmented matrix* of the system by putting the coefficient matrix and the column vector of constant terms next to each other in a single matrix like this::

$$\left( \begin{array}{cccc|c} 2 & -4 & 3 & \vdots & 1 \\ 1 & 1 & 1 & \vdots & 3 \\ 3 & -1 & -1 & \vdots & 1 \end{array} \right).$$

Notice the vertical line of dots separating the two component matrices. Sometimes this separating line is drawn as a solid vertical line. This augmented matrix contains all of the numbers that we need to work with in solving our system of equations.

We have already gone through the steps of solving the system of linear equations. Let's go through those steps again, working just with the matrix. Our first step was to multiply the first equation by  $\frac{1}{2}$ . Therefore we multiply the first row of the matrix by  $\frac{1}{2}$ :

$$\left( \begin{array}{cccc|c} 1 & -2 & \frac{3}{2} & \vdots & \frac{1}{2} \\ 1 & 1 & 1 & \vdots & 3 \\ 3 & -1 & -1 & \vdots & 1 \end{array} \right)$$

Next we subtracted equation 1 from equation 2, so we now subtract row 1 of the matrix from row 2:

$$\left( \begin{array}{cccc|c} 1 & -2 & \frac{3}{2} & \vdots & \frac{1}{2} \\ 0 & 3 & -\frac{1}{2} & \vdots & \frac{5}{2} \\ 3 & -1 & -1 & \vdots & 1 \end{array} \right)$$

Finally we added  $-3$  times equation 1 to equation 3, so working with the matrix, we add  $-3$  times row 1 to row 3:

$$\begin{pmatrix} 1 & -2 & \frac{3}{2} & \vdots & \frac{1}{2} \\ 0 & 3 & -\frac{1}{2} & \vdots & \frac{5}{2} \\ 0 & 5 & -\frac{11}{2} & \vdots & -\frac{1}{2} \end{pmatrix}$$

At this point, our system of linear equations had an  $x$  term with coefficient 1 in the first equation, and we had eliminated  $x$  from the other two equations. This means that in the first column of our matrix (the column in which we put the coefficients of  $x$ ), the first entry is 1 and all the other entries are zero. Next, working with the equations, we made the coefficient of  $y$  in the second equation equal to 1, and eliminated  $y$  from the other two equations. In terms of the matrix, this means we want the second column to have a 1 in row 2 and a zero in the other rows.

With the system of linear equations, we multiplied equations by non-zero constants, and added multiples of one equation to another. Working with the matrix, we will multiply rows by non-zero constants, and add multiples of one row to another. See if you can follow the steps:

$$\begin{pmatrix} 1 & -2 & \frac{3}{2} & \vdots & \frac{1}{2} \\ 0 & 1 & -\frac{1}{6} & \vdots & \frac{5}{6} \\ 0 & 5 & -\frac{11}{2} & \vdots & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \frac{7}{6} & \vdots & \frac{13}{6} \\ 0 & 1 & -\frac{1}{6} & \vdots & \frac{5}{6} \\ 0 & 5 & -\frac{11}{2} & \vdots & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \frac{7}{6} & \vdots & \frac{13}{6} \\ 0 & 1 & -\frac{1}{6} & \vdots & \frac{5}{6} \\ 0 & 0 & -\frac{28}{6} & \vdots & -\frac{28}{6} \end{pmatrix}$$

First we multiplied the second row by  $\frac{1}{3}$ . Then we added 2 times the second row to the first row. Then we added  $-5$  times the second row to the third row. Here are the operations that finish the job:



$$\begin{pmatrix} 1 & 0 & \frac{7}{6} & \vdots & \frac{13}{6} \\ 0 & 1 & -\frac{1}{6} & \vdots & \frac{5}{6} \\ 0 & 0 & 1 & \vdots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & -\frac{1}{6} & \vdots & \frac{5}{6} \\ 0 & 0 & 1 & \vdots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 1 \end{pmatrix}$$

Finally, we need to go from this matrix back to a system of linear equations. Remember that each row of the matrix corresponds to an equation. The columns contain, in order, the coefficients of  $x$ , the coefficients of  $y$ , the coefficients of  $z$ , and the constant terms. (Notice that it is *important* to keep track of which column corresponds to which variable.) So our new system (equivalent to our old one) is

$$\begin{aligned} 1x + 0y + 0z &= 1 \\ 0x + 1y + 0z &= 1 \\ 0x + 0y + 1z &= 1 \end{aligned}$$

$$x = 1$$

$$y = 1$$

$$z = 1$$

## 3.2 Row-Reduction of Matrices

In working with the matrix in the last section, we were allowed to do with the rows the same things we are allowed to do with the individual equations in a system of equations:

1. Multiply any row by a non-zero constant.
2. Add a constant multiple of any row to any other row.
3. Interchange any two rows.

These three operations to a matrix are called **elementary row operations**. These operations convert the augmented matrix of a system of equations to the augmented matrix of an equivalent system of equations.

Notice that in doing this conversion of the original matrix to an equivalent one, we are trying to obtain a matrix with a special form: one that corresponds to a system of linear equations whose solution is easy to write down. We actually wanted a system that looked like

$$x = \textit{number}_1$$

$$y = \textit{number}_2$$

$$z = \textit{number}_3$$

Of course, we can't always get a system like this, but we can always get a system whose solution is easy to write down. To do this, we put the augmented matrix of the system into a special form.

To describe this desired matrix form, we need one piece of terminology: The **leading entry** of a row of a matrix is the first entry in that row that is not 0. The form into which we are trying to put our matrix is a form satisfying these three conditions:

1. The leading entry of every row is 1. (An exception: A row may consist entirely of zeroes, and therefore have no leading entry.)
2. If a column contains the leading entry of any row, then every other entry in that column is 0.
3. As we go down the rows, the positions of the leading entries move from left to right. (Also, any row consisting entirely of zeroes comes after all the rows with leading entries.)

Such a matrix is said to be in **row reduced** form, or **row echelon** form and the process of using elementary row operations to put a matrix into row

echelon form is called **row reduction**. These definitions will be made more clear with an example.

**Example:** A system of linear equations in variables  $x$ ,  $y$  and  $z$  has the following augmented matrix.

$$\left( \begin{array}{ccc|c} 2 & 2 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \end{array} \right)$$

Determine whether the matrix is in row echelon form. If not, use row reduction to put it into row echelon form, and give the solution to the system.

**Solution:** This matrix is not in row echelon form. In fact, it violates all three criteria for being in row echelon form: The leading entry in row 1 is not 1. Column 2 contains the leading entry of row 3, but not all its other entries are 0 (it has a 2 in row 1.) Going down from row 2 to row 3, the positions of the leading entries move from right to left (column 3 to column 2) instead of left to right. We can put the matrix in row echelon form by the following sequence of operations:

- (1.) Multiply row 1 by  $\frac{1}{2}$ :

$$\left( \begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \end{array} \right)$$

- (2.) Interchange rows 2 and 3:

$$\left( \begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

- (3.) Add  $-1$  times row 2 to row 1:

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

This matrix is now in row echelon form. We write down the corresponding system of linear equations:

$$x = 1$$

$$y = 2$$

$$z = 3.$$

This system is already “solved”, although if we prefer, we can rewrite the solution in column vector form like this:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

### 3.3 Another example

**Example:** Find all solutions to the system of equations

$$y + z = 0$$

$$2x - y - z = 2$$

$$2x + 2y + 2z = 2$$

**Solution:**

First we write the augmented matrix of our system

$$\begin{pmatrix} 0 & 1 & 1 & \vdots & 0 \\ 2 & -1 & -1 & \vdots & 2 \\ 2 & 2 & 2 & \vdots & 2 \end{pmatrix}$$

Now we use elementary row operations to put this into row echelon form. Our first sequence of operations will be: Interchange rows 1 and 2. Multiply the new row 1 by  $\frac{1}{2}$ . Add  $-2$  times this new row 1 to row 3. The result is

$$\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & \vdots & 1 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 3 & 3 & \vdots & 0 \end{pmatrix}$$

That makes the first column into

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which is what we are after. Now we try to make the second column into

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

We are trying to do this *without* messing up what we have done with the first column. So we can interchange any two rows *except* row 1, multiply any row *except* row 1 by a constant, and add multiples of any row *except* row 1 to any other row (*including* row 1.) The sequence of operations that does the job is: Add  $\frac{1}{2}$  times row 2 to row 1. Then add  $-3$  times row 2 to row 3. This gives the new matrix

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

Next we would try to make the third column into

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

However, there is no way we can do that without interchanging rows 2 and 3 (to put a non-zero entry into row 3, column 3), and that would mess up what we have already done with column 2. In fact, our matrix is *already* in row echelon form, so we are done. Our system of equations has been converted to

$$\begin{aligned} x &= 1 \\ y + z &= 0 \\ 0 &= 0 \end{aligned}$$

The point of row echelon form is that the solutions to the system are easy to write down, and this is true. The first and second equations determine the values of  $x$  and  $y$ , and the third one  $0 = 0$ , can be ignored. So we have for our solutions:

$$\begin{aligned}x &= 1 \\y &= -z \\z &= \text{anything}\end{aligned}$$

### 3.4 Expressing Solutions

An alternative way of describing the collection of solutions in the previous example is to introduce a **parameter**  $t$ , and say solutions to the system of equations are any  $x, y, z$  of the form

$$\begin{aligned}x &= 1 \\y &= -t \\z &= t.\end{aligned}$$

A collection of solutions written in this way is called a **parametrized family** of solutions. Essentially, we have introduced the parameter  $t$  to stand in for “anything”. The meaning of this is that, for any particular choice of  $t$ , you get a particular choice for  $x, y$ , and  $z$  that satisfies the original system of equations. (That is,  $t$  can be anything). Furthermore, any solution to the original system can be gotten by taking the right value of  $t$ . In this case, choosing  $t = 2$  gives one solution

$$\begin{aligned}x &= 1 \\y &= -2 \\z &= 2\end{aligned}$$

and choosing  $t = -\frac{1}{2}$  gives a different solution

$$\begin{aligned}x &= 1 \\y &= \frac{1}{2}\end{aligned}$$

$$z = -\frac{1}{2}.$$

You might recognize this parametrized family of solutions as being scalar parametric equations for a line in three dimensions. This is exactly right. Our original system of linear equations consisted of the equations for three planes. Solving that system is finding the intersection of the three planes. These three planes intersect in a line. From the scalar parametric equations for that line, we can write down a vector parametric equation,

$$\vec{r} = \hat{i} + t(-\hat{j} + \hat{k}).$$

## 4 Representing Solutions Geometrically

### 4.1 Solutions in two dimensions

The solutions to a system of linear equations in two variables, such as the system

$$\begin{aligned}x + 2y &= 25 \\ 3x + 4y &= 65.\end{aligned}$$

can be thought of as a collection of points in the plane, in this case in the  $x, y$ -plane. We know a single equation of the form

$$ax + by = c$$

is the equation of a line in the plane; that is, the collection of solutions to that equation, represented as a collection of points in the plane, forms a line. The two equations in this system are equations of the two lines pictured in Figure 1.

Solutions to this system of equations are points that are solutions to each of the individual equations, that is, points that lie on both lines. In this case, there is exactly one such point,  $(x, y) = (15, 5)$ . This is what we usually expect for the solution to a system of two equations in two unknowns; the intersection of two lines, a single point or a unique solution.

This is not the only possibility, however. The two lines could be parallel, in which case there are no solutions to the system, because the two lines do not intersect. Or the two lines could coincide (the two equations could actually be equations for the same line), in which case there are infinitely

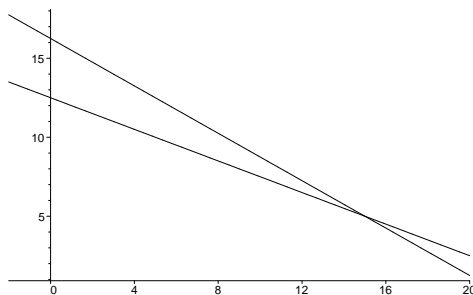


Figure 1: Solution to a System of Linear Equations

many solutions, namely all the points on that line. The same possibilities (no solutions, one unique solution, or infinitely many solutions forming a line) occur with any number of equations in two variables.

These, however, are the only possibilities. For example, it is not possible to have two (or three, or more) lines intersecting in exactly two points (because any lines passing through the same two points must be the same line), so it is not possible to have a system of linear equations in two variables with exactly two solutions.

## 4.2 Solutions in three dimensions

A similar geometric analysis occurs in three dimensions. We know that just as the equation

$$ax + by = c$$

represents a line in the plane, the equation

$$ax + by + cz = d$$

represents a plane in three-dimensional space. So the set of solutions to a system of linear equations in three variables, viewed as a collection of points in space, is the intersection of the planes represented by those equations. There are four possibilities for these solutions corresponding to their geometric representations:

1. The planes do not intersect, and there are no solutions.
2. The planes intersect in a single point, and there is a unique solution.



3. The planes intersect in a line, and there is a parametrized family of solutions, of the form

$$x = a_1 + tb_1$$

$$y = a_2 + tb_2$$

$$z = a_3 + tb_3$$

where  $t$  is a parameter.

4. The planes coincide, and there is a parametrized family of solutions, of the form

$$x = a_1 + tb_1 + sc_1$$

$$y = a_2 + tb_2 + sc_2$$

$$z = a_3 + tb_3 + sc_3$$

where  $s$  and  $t$  are two different parameters. This two-parameter family forms a two-dimensional collection of solutions, which geometrically is a plane.

**Example:** In Section 3.3; the three planes with equations

$$y + z = 0$$

$$2x - y - z = 2$$

$$2x + 2y + 2z = 2$$

intersect in the line described by the equations

$$x = 1$$

$$y = -t$$

$$z = t$$

as in Figure 2.

**Example:** Let's find the intersection of the three planes with equations

$$y + z = 0$$

$$2x - y - z = 4$$

$$2x + 2y + 2z = 2$$

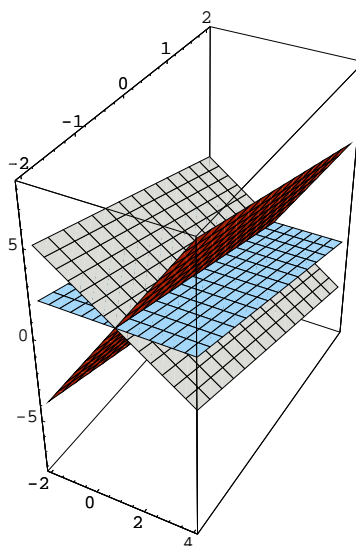


Figure 2: Solution to a System of Linear Equations

The augmented matrix of this system of equations is

$$\left( \begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 2 & -1 & -1 & 4 \\ 2 & 2 & 2 & 2 \end{array} \right)$$

Put into row echelon form, this becomes

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

which gives the system of equations

$$\begin{aligned} x &= 0 \\ y + z &= 0 \\ 0 &= 1 \end{aligned}$$

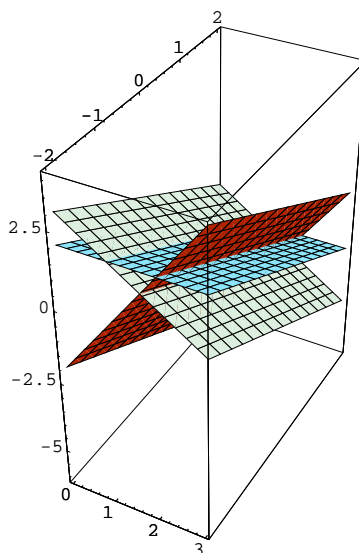


Figure 3: Solution to a System of Linear Equations

This last equation tells us that the system has no solutions. These planes do not intersect, as in Figure 3. Note: You can see from the equations of the planes that these three planes are parallel to the three planes in the previous example.

**Example:** The system of linear equations

$$x + y + z = 1$$

$$2x + 2y + 2z = 2$$

$$-x - y - z = -1$$

yields the equivalent system (as you should verify for yourself as practice)

$$x + y + z = 1$$

$$0 = 0$$

$$0 = 0.$$

We can ignore the last two equations. The first one determines the value of  $x$  in terms of  $y$  and  $z$ :

$$x = 1 - y - z$$

The leftover variables  $y$  and  $z$  can be anything, so we can assign them parameters  $t$  and  $s$ , to get a parametrized family of solutions

$$x = 1 - t - s$$

$$y = t$$

$$z = s$$

where  $s$  and  $t$  can be anything. (Note: It is important that we do NOT assign  $y$  and  $z$  the same parameter. If we said  $y = t$  and  $z = t$ , we would be saying that  $y = z$ ; but our system of equations does not say that.) These are parametrized scalar equations for a plane in three dimensions.

### 4.3 Higher dimensions

A system of linear equations in  $n$  different variables,  $x_1, x_2, \dots, x_n$ , can also be analyzed geometrically. A linear equation

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

determines an  $(n - 1)$ -dimensional collection of solutions (sometimes referred to as a “hyperplane”.) Two such  $(n - 1)$ -dimensional hyperplanes (unless they happen to be identical) intersect in an  $(n - 2)$ (or fewer)-dimensional collection of solutions, etc.

This means that a system of linear equations in  $n$  variables may have no solutions, a single unique solution, or infinitely many solutions. In the case of infinitely many solutions, the solutions form a parametrized family, which may be expressed using anywhere from 1 to  $n - 1$  parameters; this corresponds to the geometric dimension of the solution set.

**Example:** The system of equations

$$v + w + x + y + z = 1$$

$$v - w + x - y + z = 3$$

$$3v - w + 3x - y + 3z = 7$$

has the following three-parameter family of solutions:

$$v = 2 - r - t$$

$$w = -1 - s$$

$$x = r$$

$$y = s$$

$$z = t$$

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**Exercise 1** For each of the following systems of linear equations, sketch on the same graph the lines described by each equation, and determine whether the system has no solutions, one solution, or infinitely many solutions.

$$x + y = 1$$

$$3x + 3y = -3$$

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$$x + 2y = 2$$

$$2x + 4y = 4$$

---

$$3x - y = 3$$

$$x - y = 3$$

---

$$x + y = 1$$

$$x - y = 1$$

$$2y - x = 2$$

---

$$x + y = 1$$

$$x - y = -1$$

$$2y - x = 2$$

---

$$x + y = 1$$

$$x - y = 1$$

$$y - x = 1$$

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**Exercise 2** Determine which of the following matrices are in row echelon form. For those which are not, give a sequence of elementary row operations that will put them in row echelon form.

$$\begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

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**Exercise 3** Solve the system of linear equations

$$w + x + y + z = 2$$

$$w - x - y = 0$$

$$3w + y + 2z = 5$$

$$6w + x + 2y + 4z = 9$$

and give the solution in parametric form.

One solution to this system of equations is  $w = 1$ ,  $x = -1$ ,  $y = 2$ , and  $z = 0$ . What choice of parameters in your solution gives this particular solution?

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**Exercise 4** *Solve the following systems of linear equations by row-reducing their augmented matrices.*

$$x + y = 1$$

$$x - y = 3$$

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$$x + y - z = 1$$

$$2x + 3y + z = 6$$

$$x - y + 2z = 2$$

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$$y + z = 0$$

$$2x - y - z = 4$$

$$2x + 2y + 2z = 2$$

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$$v + w - x = 1$$

$$2v - w = 1$$

$$3v - x = 2$$

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**Exercise 5** *Explain why a system of two linear equations in three variables may have no solutions or infinitely many solutions, but will never have exactly one solution.*

*Which outcomes (no solutions, one solution, infinitely many solutions) can possibly occur in systems of four linear equations in three variables?*