FINAL REVIEW SHEET, MATH 8, FALL 2010

This sheet contains a quick summary of definitions, tests, and other facts which might be useful for the final exam. There are no examples on this sheet, and is only intended to be a convenient reference where you can see all the basic ideas of the last part of the term in a single place. The following lists may not be comprehensive – it is possible that I forgot to include certain definitions or theorems! For material that was covered on first two midterms, see the previous two review sheets. There will be questions on topics from the first two-thirds of the class!

1. Vector valued functions

- Know what vector-valued functions are, how to differentiate them, and how to integrate them.
- Know how to calculate tangent lines to curves, and the interpretation of derivatives as velocity and acceleration.
- Know the arc length formula:

$$\int_{t_1}^{t_2} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

  for a vector valued function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ from t_1 to t_2. Also know the obvious generalization of this formula to vector-valued functions in $\mathbb{R}^n$.

2. Functions of several variables

- Know how to determine the domain of a function of several variables.
- Know how to show that limits of functions of several variables do not exist, at least in simple examples.
- Know how to calculate partial derivatives.
- Applications of partial derivatives: know how to use partial derivatives to calculate tangent planes to surfaces $z = f(x, y)$, how to calculate linearizations, and how to calculate differentials. The equation for a tangent plane to $z = f(x, y)$ at (x_0, y_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

  The linearization of $f(x, y)$ at (x_0, y_0) is obtained by solving for z in this expression.
- Know the chain rule for functions of several variables. For instance, if $x = x(t), y = y(t)$, and $z = f(x, y)$, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

  Also know the generalizations for functions of more than two variables, as well as functions $x = x(s, t), y = y(s, t)$ where x, y may be functions of more than one variable.
- Directional derivatives: know the definition and the geometric significance. Remember to use unit vectors when calculating directional derivatives!
- Gradient vector: know the definition, $\nabla f(x, y) = \langle f_x, f_y \rangle$. Know how to use this to calculate directional derivatives: $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$.

- Applications of this formula for directional derivatives: know that the gradient points in the direction of maximal increase, and that the length of the gradient is the rate of change in this direction.
- For curves $f(x, y) = 0$ or surfaces $f(x, y, z) = 0$, know that ∇f gives normal vectors to the tangent plane and tangent planes, respectively, of these two graphs.
- Critical points: critical points of $f(x, y)$ are points at which $f_x, f_y = 0$, or at least one of f_x, f_y does not exist. These are *candidate* (not certain!) local extrema.
- Second derivative test: if f_{xx}, f_{xy}, f_{yy} all exist, are continuous, then second derivative test can be used to possibly determine the nature of critical points. Namely, let $D = f_{xx}f_{yy} - f_{xy}^2$, and (a, b) be a point at which $f_x(a, b) = f_y(a, b) = 0$, then
  - If $D > 0, f_{xx}(a, b) < 0$, then (a, b) is a local maximum,
  - if $D > 0, f_{xx}(a, b) > 0$, then (a, b) is a local minimum,
  - if $D < 0$, then (a, b) is a saddle point, and
  - if $D = 0$, the test is inconclusive.
- Lagrange multipliers: if you want to find the absolute maximum or minimum of $f(x, y)$ where x, y are restricted to satisfy the equation $g(x, y) = 0$, and $\nabla g \neq \mathbf{0}$ everywhere on $g(x, y) = 0$, then you can use the method of Lagrange multipliers, which says that if absolute extrema exist, then they occur at points which satisfy $\nabla f = \lambda \nabla g$, for some real number λ.
- The solutions to this 'Lagrange multiplier system' are merely candidates for absolute extrema, not necessarily absolute extrema. It is possible for no absolute extrema to exist, but if $g(x, y) = 0$ defines a region which is bounded (that is, none of its points goes off to infinity), then absolute extrema of $f(x, y)$ will exist.
- When solving the 'Lagrange multiplier system' be very sure you are actually finding all solutions. In particular, when you divide by an expression, make sure to either verify that the expression is never 0, or if it is, to separately analyze what happens when that expression is 0. (For instance, if you see the equation $x = \lambda x$, it is not quite correct to conclude that $\lambda = 1$, because it is possible that $x = 0$, which would then impose no restriction on λ. So you will have to separately analyze the $\lambda = 1$ and $x = 0$ cases.)
- Also know how to use Lagrange multipliers for functions of more than two variables. You do not need to know how to handle multiple constraints, though.
- Finding absolute extrema on some closed region D, like a disc $x^2 + y^2 \leq 1$: this is a two-step problem. First find critical points on the interior of D, and then find the absolute extrema on the boundary of D (if they exist). If absolute extrema exist, they will either appear on the boundary or at a critical point. Again, absolute extrema may not exist if D is a region which is not bounded, but if D is bounded, absolute extrema will exist. (For instance, if D is the region $x \geq 0$, absolute extrema of some function $f(x, y)$ on this region may not exist, but if D is the region $-1 \leq x, y \leq 1$, which is bounded, absolute extrema of any function $f(x, y)$ continuous on this region will exist.)

3. Tips for preparing

The following suggestions may or may not help you prepare for the test. (This is identical advice to that given on the first two review sheets.)

- Take a look at all your old homework assignments, and learn how to solve them all by yourself. Many people either received assistance or took many attempts to correctly answer questions – at this point, you should be able to solve every old homework problem, correctly, on your own.
- For additional practice problems, try solving random questions from the text.

- Make sure you are precise with arithmetic and algebra. Being able to do both these accurately, by hand, is an important skill to cultivate.
- Make sure you are well-rested and not hungry during the exam. You may also want to bring a light coat, in case the room is too cold. Also make sure you bring enough pencils or pens to the test site.
- This isn't a tip, but PLEASE turn off all cell phones and other portable electronic devices during the exam. Even phones on vibrate settings are a nuisance to nearby test takers.
- Try to keep the time limit in mind as you take the exam. Do not spend too much time obsessing about a small number of problems if it will be detrimental to your performance on the remaining problems.
- Show all your work, and try to be organized and neat when you write your solutions. This will make it easier for the graders to follow your line of thought, and may help you earn (more) partial credit if you cannot completely solve a question correctly.