MATH 8 CLASS 8 NOTES, 10/8/2010

In the last class, we examined a special type of series called a power series, which has the form $\sum c_n(x-a)^n$. We found that the ratio test was a useful tool for determining the values of x for which such a series converges. In addition, we found that the set of points for which such a power series converges is always an interval, centered around a, with radius R. Convergence at the points $x = a - R, a + R$ has to be checked manually as the ratio test is inconclusive at those points.

In summary, we were able to determine the values of x for which a power series converges, and hence defines a function. We can instead ask an opposite question: given a function $f(x)$, can we find a power series which represents $f(x)$, at least for some values of x? Although this seems like a simple question to ask given what we have already done, it provides a powerful method for approximating a large class of functions and its answer has proven useful in mathematics, science, and engineering.

1. A special case: functions as geometric series

We will begin by examining some special functions for which we can easily express them as power series using techniques we already know. Although these methods are far too simple to handle all the different functions we are interested in, they do illustrate important parts of the theory without introducing a tremendous number of details.

Consider the function $f(x) = 1/(1-x)$. We know that this is equal to the geometric series $1 + x + x^2 + x^3 + \dots$ when $|x| < 1$. When $|x| \geq 1, f(x)$ is still defined (except for $x = 1$), but the geometric series $1 + x + \dots$ makes no sense there since it does not converge. We see that the function $1/(1-x)$ can be expressed as a power series centered at $x = 0$, with radius of convergence $R = 1$. This brings up the first point about expressing a function as a power series:

Given a function f, it may be possible to express f as a power series centered at some point $x = a$. However, that power series need not converge on the same set of points that f is defined on, so that particular power series expression will be invalid outside the set of points on which it converges.

There are methods which allow you to take a function $f(x)$ for which you have a power series expansion and obtain power series expansions for functions of the form $xf(x)$ or $f(bx)$, where $b \neq 0$. Let us illustrate this in the case of the function $f(x) = 1/(1-x)$.

Examples.

- Find a power series expansion for $x^2/(1-x)$ around the point $x = 0$. We already know that $1/(1-x) = 1 + x + x^2 + \dots$ for $|x| < 1$. Therefore,
$$\frac{x^2}{1-x} = x^2 + x^3 + x^4 + \dots.$$
We obtain this equality by multiplying the power series expansion $1 + x + x^2 + \dots = 1/(1-x)$ by x^2. This power series still converges for $|x| < 1$, so the radius of convergence is unaltered by this particular operation. In general, you should always keep track of the interval of convergence for the different power series you obtain.
- Find a power series expansion for $1/(1+3x)$ around the point $x = 0$. Again, we start from the expansion $1/(1-x) = 1 + x + x^2 + \dots$. This time, we make the substitution $-3x$ for x, and find that

$$\frac{1}{1+3x} = 1 - 3x + 9x^2 - 27x^3 + 81x^4 + \dots.$$

Notice that this time, the radius of convergence changes from $R = 1$ to $R = 1/3$, since the new power series is a geometric series with ratio $3x$. Alternatively, if one recognizes $1/(1+3x)$ as the expression for a geometric series with $a = 1, r = -3x$, one can also directly obtain the power series expansion for $1/(1+3x)$.

- Find a power series expansion for $x/(2+x)$ around the point $x = 0$. We start by considering the simpler problem of finding a power series expansion for $1/(2+x)$. We want to get this into the form $1/(1-bx)$, for some $b \neq 0$, so we divide both the numerator and denominator by 2 to obtain

$$\frac{1}{2+x} = \frac{1/2}{1+(x/2)} = \frac{1}{2}\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \dots\right).$$

Now we multiply both sides by x to obtain

$$\frac{x}{2+x} = \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{8} - \frac{x^4}{16} + \dots.$$

The radius of convergence of this power series is $R = 2$. Again, one could also obtain this power series expansion directly by recognizing

$$\frac{x}{2+x} = \frac{(x/2)}{1-(-x/2)}$$

as a geometric series with initial term $a = x/2$ and ratio $r = -x/2$.

- Find a power series expansion for $1/(1+x^2)$. If we substitute $-x^2$ for x in the expansion for $1/(1-x)$, we obtain

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots.$$

This power series still has radius of convergence $R = 1$. We could also have obtained this power series expansion directly by recognizing $1/(1+x^2)$ as the expression for a geometric series with $a = 1, r = -x^2$.

As the above examples show, we can obtain the power series representation of a surprising number of functions using nothing more than the geometric series $1+x+x^2+\dots$ and simple substitution or term-by-term multiplication rules. However, we still want to broaden the types of functions we can find power series expansions for.

2. Differentiating and Integrating power series

A power series $\sum c_n x^n$ (we center the power series around $a = 0$ for convenience) defines a function in some interval $(-R, R)$, and may or may not be defined at the endpoints of this interval. We can then ask: is it possible to differentiate and integrate this function in this interval? If so, how can we represent the derivative and integral of a power series in terms of the original power series, and how does the radius of convergence change?

Notice that some simple considerations already suggest an answer. If we believe that a power series is an infinite-degree generalization of a polynomial, then we might expect that the way we differentiate or integrate a power series is just like how we differentiate or integrate a polynomial. For example, the derivative of the polynomial $1 + x + x^2 + x^3$ is $1 + 2x + 3x^2$, which we know using the 'power rule' from Calculus. Then it might stand to reason that the series

$$1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n$$

is the derivative of $1 + x + x^2 + \ldots$. As a matter of fact, the following theorem (which we will not prove) says that this is true:

Theorem. Let $\sum_{n=0}^{\infty} c_n x^n$ be a power series with radius of convergence R. Then the series

$$\begin{aligned} c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots &= \sum_{n=0}^{\infty} (n+1)c_{n+1}x^n, \\ c_0x + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3 + \ldots &= \sum_{n=0}^{\infty} \frac{c_n}{n+1}x^{n+1} \end{aligned}$$

are the derivative and integral of the original series, respectively. (Each is obtained via term-by-term differentiation or integration). Furthermore, both these new series also have radius of convergence R. The convergence of the series at the endpoints of the interval of convergence may change after differentiation or integration, though.

Examples.

- Find a power series expansion for $\arctan x$ centered around $x = 0$. At first glance, this problem might look intractable, given what we know. However, if we remember that $\arctan x = \int 1/(1 + x^2)dx$, and that we found the power series expansion for $1/(1 + x^2) = 1 - x^2 + x^4 - x^6 + \ldots$ in the previous section, we find that
$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$
is a power series expansion for $\arctan x$ centered around $x = 0$. One checks that the radius of convergence of this power series is still $R = 1$, using the ratio test. Notice that the power series for $\arctan x$ is no longer geometric.
- Find a power series expansion for $1/(1 - x)^2$ centered around $x = 0$. Again, this problem is inaccessible using only the techniques of the previous section. In particular, we cannot recognize this series as representing the value of a geometric series. However, if we remember that the derivative of $1/(1-x)$ is $1/(1-x)^2$, then we find that
$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n.$$
This power series still has radius of convergence $R = 1$, which one can check using the ratio test.
- Let's think about how to find a power series expansion for e^x. The distinguishing characteristic of e^x is that it is the only function (up to a constant multiple) which is its own derivative. So we want to find a power series $\sum c_n x^n$ such that its derivative is equal to itself. The derivative of $\sum_{n=0}^{\infty} c_n x^n$ is
$$c_1 + 2c_2x + \ldots = \sum_{n=1}^{\infty} nc_nx^{n-1} = \sum_{n=0}^{\infty} (n+1)c_{n+1}x^n.$$
If we want this to equal the original series $\sum c_n x^n$, then we should have $c_n = (n+1)c_{n+1}$, for all n, or $c_{n+1} = c_n/(n+1)$. Suppose $c_0 = 1$. Then $c_1 = 1/1 =$

$1, c_2 = 1/2, c_3 = 1/2(3) = 1/6, c_4 = 1/6(4) = 1/24, \ldots$. The pattern is clear: we need $c_n = 1/n!$. Therefore, we know that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is a multiple of e^x. To actually check that this equals e^x, plug in the value $x = 0$. We know that $e^0 = 1$, and this power series also has value 1 at $x = 0$, so this is the power series of e^x. From the two previous examples we've done in this class, we also know that this power series is convergent for all real x, so this power series really does equal e^x for all real x. In the next class we will see another way to determine the power series representation of e^x.

By the way, an interesting consequence of this power series representation is that it gives a formula for e in terms of an infinite series:

$$e^1 = e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots.$$

If one wants to use this series representation to obtain a numerical approximation to e, then one is led to the natural question of determining how to estimate the error when using partial sums as approximations to the actual value of the series. We will briefly discuss this in the next two classes, although unfortunately the answer is more complicated than the corresponding answers for series passing the integral test or the alternating series test.