MATH 8 CLASS 7 NOTES, 10/6/2010

So far, we've examined sequences and series, and spent a lot of time determining whether series converge or diverge. In particular, we have tests for geometric series, the nth term test for divergence, the integral test, the comparison tests, the alternating series test, the p-series test, and the ratio test. For series passing the integral test or the alternating series test we even have a way to estimate the error when we approximate the value of such a series with partial sums. Our current goal is to relate the idea of series, which already can be seen to be related to calculus, to the functions which were studied so extensively in Calculus I.

1. Power Series

We will consider a special type of series, which can be thought of as a function of a real variable x. A series of the form

$$\sum c_n x^n, \text{or more generally } \sum c_n (x-a)^n$$

is called a power series. The x is to be thought of as a variable, while the c_n is some sequence of numbers. In the case of the latter series, a is some fixed real number, and the first series is just the special case $a = 0$ of the second series. The index of summation is intentionally left vague, but will always start with at least $n = 0$. (In particular, we do not allow negative powers of x in a power series.) (When $n = 0$, we take the 0th term to be c_0. In particular, when $x = 0$, the power series $\sum c_n x^n$ is equal to c_0.)

The partial sums of a power series are polynomials in x, of successively higher degrees. A question we will study extensively is for what values of x a given power series converges. For example, any power series $\sum c_n x^n$ converges for $x = 0$, since every term except possibly c_0 disappears. Before stating the general facts that are true, let us look at several model examples:

Examples.

- $\sum_{n=0}^{\infty} x^n/n!$. We saw this example at the end of last class, and found that it converges for all x by the ratio test.
- $\sum_{n=0}^{\infty} x^n$. This is the power series with $a_n = 1$ for all n. As a matter of fact, this is just a geometric series with $a = 1$ and $r = x$. Therefore, this series converges when $|x| < 1$ and diverges when $|x| \geq 1$. In this example, we know that the value of the series is $1/(1-x)$ when it does converge.
- $\sum_{n=0}^{\infty} n!x^n$. We use the ratio test on this series, and find that the ratio of consecutive terms is
$$\frac{x^{n+1}(n+1)!}{x^n n!} = x(n+1)$$
If $x \neq 0$, then this ratio diverges to infinity, and hence the power series diverges by the ratio test. So this power series only converges for $x = 0$.
- $\sum_{n=1}^{\infty} x^n/n$. If we use the ratio test on this series, we find the ratio of consecutive terms is
$$\frac{x^{n+1}n}{x^n(n+1)} = \frac{xn}{n+1}$$
Therefore, the ratio test tells us this series converges when $|x| < 1$, and diverges when $|x| > 1$. Notice that the ratio test is inconclusive when $|x| = 1$. Therefore, for

these two points, we need to check the convergence or divergence of the power series using another method. For $x = 1$ the power series becomes $\sum 1/n$, which is the harmonic series, so it diverges. For $x = -1$, the power series becomes $\sum(-1)^n/n$, which is the alternating harmonic series, so this series converges. Therefore, the power series converges for all x in the interval $[-1, 1)$.

In these examples, we saw cases where power series only converged for $x = 0$, cases where the series converged for all x, cases where the series converged for all x in some open interval $(-1, 1)$, and cases where the series converged for all x in a half open interval. The common theme for all these cases is that we could find an *interval of convergence*; ie, we could describe the set of all x for which these power series converged using an interval. (We include the cases of $x = 0$ and all real x as intervals.) As a matter of fact, this holds true for any power series:

Theorem. Let $\sum c_n x^n$ be any power series. Then there exists a number R, called the radius of convergence, such that the series diverges for any x with $|x| > R$, and converges for any x with $|x| < R$. We include the cases $R = 0, \infty$, as representing the cases where $x = 0$ and all real x are the sets of convergence, respectively.

This theorem is not entirely obvious nor is it particularly easy to show, but in each example we see we will find it relatively easy to compute an interval of convergence. This theorem obviously holds true for more general power series $\sum a_n(x - c)^n$, except that instead of looking at $|x| < R$ as a set of convergence, we look instead at the set $|x - c| < R$ as a set of convergence. Note that this theorem says nothing about whether the power series in question converges for the two values of x satisfying $|x| = R$. In general, we cannot say anything about convergence at these endpoints of the interval, since there are cases of series where the series diverges at both endpoints, or converges at both endpoints, or converges at one and diverges at the other. Finally, this theorem really does say that the set of points for which a power series converges is actually an interval; that is, a set of the form $(a, b), [a, b], etc.$, with the cases of $x = 0$ and all x included.

Let us look at some slightly more complicated examples. In all these instances, the test of choice will be the ratio test, although in some instances, other tests (alternating series test, nth term test) could also be used. When a question asks for the interval of convergence, you not only need to calculate the radius of convergence, but also individually test for convergence at the endpoints of the interval.

Example.

- Determine the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1} = 1 - x/3 + x^2/5 - x^3/7 + \ldots$. We apply the ratio test; the ratio of consecutive terms is
$$\left|\frac{x^{n+1}(2n+1)}{x^n(2n+3)}\right| = \left|\frac{x(2n+1)}{2n+3}\right|$$
This has limit $|x|$ as $n \to \infty$, so the radius of convergence of the series is $R = 1$. We now test for convergence at the endpoints of the interval. When $x = -1$, we end up with the series $\sum_{n=0} 1/(2n + 1)$. Notice that this looks a lot like a harmonic series – as a matter of fact, if we multiply the series by $1/2$ and add it back to the original series, we obtain the harmonic series. This shows that this series diverges. At $x = 1$, we find that we have the alternating series $\sum_{n=0}^{\infty}(-1)^n/(2n + 1)$, which converges by the alternating series test. Therefore, this power series has interval of convergence $(-1, 1]$ and radius of convergence $R = 1$.

- Determine the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

  Use the ratio test. The ratio of consecutive terms is given by

$$\left| \frac{(-1)^{n+1} x^{2n+3}}{2n+3} \cdot \frac{2n+1}{(-1)^n x^{2n+1}} \right| = \frac{x^2(2n+1)}{2n+3}.$$

  As this tends towards the limit x^2 as $n \to \infty$, the ratio test tells us that this series converges when $x^2 < 1$ and diverges when $x^2 > 1$. So we know the radius of convergence is $R = 1$.

  Now we need to test for convergence at $x = \pm 1$, where the ratio test is inconclusive. When $x = 1$, we get the alternating series $1 - 1/3 + 1/5 - 1/7 + \dots$, which evidently passes the alternating series test. Similarly, when $x = -1$, we get the alternating series $-1 + 1/3 - 1/5 + 1/7 - \dots$, which is just the negative of the previous series, so also converges. Therefore, the interval of convergence is $[-1, 1]$.

- Determine the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{e^n (x-2)^n n}{3^n}$. Apply the ratio test; the ratio of consecutive terms is

$$\left| \frac{e^{n+1}(x-2)^{n+1}(n+1)3^n}{e^n (x-2)^n (n) 3^{n+1}} \right| = \left| \frac{e(x-2)(n+1)}{3n} \right|$$

  As $n \to \infty$, this has limit $|e(x-2)/3|$. This is < 1 precisely when $|x-2| < 3/e$, or when $2 - 3/e < x < 2 + 3/e$. We now test for convergence at the endpoints. When $x = 2 + 3/e$, we obtain the series $\sum_{n=0}^{\infty} n$, which obviously diverges, while when $x = 2 - 3/e$, we obtain the series $\sum_{n=0}^{\infty} n(-1)^n$, which also obviously diverges. Therefore, the interval of convergence is $(2 - 3/e, 2 + 3/e)$. The radius of convergence is $3/e$.

- Here is a slight variation on the above problems. Suppose we are given a power series $\sum c_n x^n$ which has radius of convergence $R = 2$. Consider the related power series $\sum c_n x^{2n}$. Does this series converge at $x = 1$? How about $x = 3/2$? What is the radius of convergence of this series?

  To answer the first two questions, plug in $x = 1, 3/2$: we get the series $\sum c_n 1^{2n} = \sum c_n 1^n$. This is just the original series evaluated at $x = 1$, which we know converges since $1 < R = 2$. Similarly, for $x = 3/2$, we get the series $\sum c_n (3/2)^{2n} = \sum c_n (9/4)^n$. This is the original series at $x = 9/4$, which diverges since $9/4 > R = 2$.

  To find the radius of convergence, we follow the idea in these two examples. Since $\sum c_n x^{2n} = \sum c_n (x^2)^n$, this series converges when $x^2 < 2$ and diverges when $x^2 > 2$. Therefore, this series has radius of convergence $R = \sqrt{2}$.

- Similar reasoning to the above applies when we want to consider a series $\sum c_n (x-a)^n$ or $\sum c_n (x/b)^n$. If $\sum c_n x^n$ has radius of convergence R, then $\sum c_n (x-a)^n$ has radius of convergence R as well (this time the interval is centered at a, not 0), while $c_n (x/b)^n$ has radius of convergence $R|b|$. In principle, one could mix and match these three ways of modifying a power series.

In summary, a power series is any series of the form $\sum c_n (x-a)^n$. The set of x for which this series converges is an interval. To determine the radius of convergence, you usually want to use the ratio test. To determine convergence at the endpoints of the interval, you will need to usually apply some other test.