MATH 8 CLASS 6 NOTES, 10/4/2010

So far, we have discussed geometric series, the nth term test for divergence, the alternating series test, and p-series. This is a pretty good list of tests, but there is one more test we will learn, which will turn out to be very useful. This test is called the ratio test and can be considered a generalization of the test for geometric series.

1. The ratio test

Suppose we have a series $\sum a_n$. The only restriction we impose on the a_n is that $a_n \neq 0$ for all n.

Theorem. (The Ratio Test). Let $\sum a_n$ be any series with nonzero terms. Consider the limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

If $L < 1$, then $\sum a_n$ converges (actually, absolutely converges). If $L > 1$, then $\sum a_n$ diverges. If $L = 1$ or L does not exist, then the ratio test is inconclusive.

You should think of the ratio test as a generalization for the test we use for geometric series. If $\lim_{n\to\infty} a_{n+1}/a_n = L$, then you can think of $\sum a_n$ as 'approximately' being a geometric series with ratio L (after all, if $\sum a_n$ actually is a geometric series with ratio L, then $a_{n+1}/a_n = L$). Then the ratio test makes intuitive sense, since a geometric series converges exactly when $|r| < 1$, and diverges when $|r| > 1$. The proof of the ratio test (which we will not write down) is actually quite simple. We simply make a direct comparison of $\sum a_n$ with a geometric series of ratio L, and it is not hard to make this idea precise.

Let's take a look at a lot of different examples of the ratio test. Pay attention to the 'shape' of the series on which the ratio test works, and the 'shape' of the series on which the ratio test does not work.

Examples.

- Consider $\sum_{n=1}^{\infty} 1/n!$. We already tested this using direct comparison to $\sum 1/n^2$, say, but this was somewhat tedious. The ratio test makes this much easier:
$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1}.$$
The limit of this ratio as $n \to \infty$ is 0, so this series converges by the ratio test.
- Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n(n+1)}$ converge or diverge? Notice that this is a slight modification of a convergent geometric series. If we try to use the ratio test on this series, we find that
$$\frac{a_{n+1}}{a_n} = \frac{n+1}{3^{n+1}(n+2)} \frac{3^n(n+1)}{n} = \frac{1}{3} \frac{(n+1)^2}{n(n+2)}.$$
As $n \to \infty$, the ratio of quadratic polynomials converges to 1, so the value of the limit of this ratio is $1/3$. Therefore, the series converges by the ratio test.
- Apply the ratio test to the series $\sum n$. (We already know this diverges by the nth term test, but we want to see what happens when we use the ratio test.) We find that $a_{n+1}/a_n = (n+1)/n$, which has limit 1. Therefore, the ratio test is inconclusive. However, we easily see that this series is divergent by other means. This example

shows that the ratio test, while being quite general, is not the definitive test for determining convergence and divergence.

- Apply the ratio test to the series $\sum 1/n^2$. (Again, we know this converges, because it is a p-series with $p > 1$.) The ratio a_{n+1}/a_n is given by
$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2}.$$
This has limit 1, so again the ratio test is inconclusive. This example shows that when the ratio test is inconclusive, the series in question could really either converge or diverge, so if the ratio tests yields an inconclusive result, we need to use some other test to determine convergence or divergence.
- Apply the ratio test to the series $1 + 1/4 + 1/2 + 1/8 + 1/4 + 1/16 + 1/8 + 1/32 + \dots$ The general pattern here is that we alternate multiplying a number by 1/4 and 2. As a matter of fact we see that this is the sum of two convergent geometric series and hence is convergent. However, if we apply the ratio test, we see that the ratio a_{n+1}/a_n oscillates between 1/4 and 2, so the ratio diverges. Therefore, the ratio test is inconclusive, and again we see that the series converges by means of other ideas.

From these examples it seems the ratio test works well when we have exponentials in n (as in a geometric series) or factorials in n. The ratio test is 'insensitive' to polynomials in n, as the two examples involving $\sum n, \sum 1/n^2$, suggest. However, this fact can work for us, since it makes testing geometric series modified by 'polynomial embellishment' feasible with the ratio test. Let's look at more complicated examples: