MATH 8 CLASS 4 NOTES, 9/29/2010

1. The Direct Comparison Test

So far, we have three types of tests for determining whether a series converges: the geometric series test (if a series is geometric, which is rather rare), the nth term test for divergence, and the integral test. The integral test has some limitations; for instance, we need the terms of a series to be monotone decreasing and the values of a function at integers which we can easily integrate.

Today, we'll learn about a pair of tests which let us translate the problem of determining the convergence or divergence of a series to understanding the convergence or divergence of a simpler, 'related' series.

The idea is a variation on the idea behind the integral test. Suppose we have two sequences, a_n, b_n, which satisfy $0 \leq a_n \leq b_n$ for all n. Then

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i,$$

so the partial sums for the series $\sum a_n$ are less than the corresponding partial sums for $\sum b_n$.

If $\sum a_n$ diverges, this means that the sequence of its partial sums diverge to infinity (notice that we are using the fact that $a_n \geq 0$), so from this we can conclude that $\sum b_n$ also diverges. In other words, what we are saying is that if $0 \leq A \leq B$, and A is very big, B is also very big.

Now suppose that $0 \leq a_n \leq b_n$, except that now we know $\sum b_n$ converges, say to L. Since all the a_n, b_n are positive, this means that

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i \leq L.$$

But then this tells us that $\sum a_n$ converges, because its sequence of partial sums is a monotonic bounded sequence (monotonic increasing since each a_i is positive, and bounded from below by 0 and above by L). Again, basically what is happening here is that we are saying if $0 \leq A \leq B$, and if B is small, then A has to be small as well.

These two arguments give us the following test:

The direct comparison test. Suppose $0 \leq a_n \leq b_n$ for all n. Then

- If $\sum a_n$ diverges, then $\sum b_n$ diverges.
- If $\sum b_n$ converges, then $\sum a_n$ converges.

It is probably worth pointing out that if $\sum a_n$ converges, you can't conclude anything about $\sum b_n$, and if $\sum b_n$ diverges, you can't conclude anything about $\sum a_n$. Let's look at a few concrete examples to see why this test can be helpful.

Examples.

- Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1.5+(-1)^n}{n}$. This series is not a p-series, nor a geometric series, nor a series we can apply the integral test to, nor a series which we can apply the nth term test to. But it looks pretty similar to the harmonic series. Let's write the first few terms of this series out: $0.5/1 + 1.5/2 + 0.5/3 + 1.5/4 + 0.5/5 + \ldots$.

  Notice that $0 \leq 0.5/n \leq (1.5+(-1)^n)/n$, regardless of what n is. This is a setup which we can use the direct comparison test, with $a_n = 0.5/n, b_n = (1.5+(-1)^n)/n$. In this case, it is the series a_n which we can easily determine the convergence or divergence of; it diverges since it is just a non-zero constant times the harmonic series. So the direct comparison test tells us that the original series is divergent as well.
- Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{(\sin n)^2}{n^3}$. The individual terms of this series are difficult to get a handle on, since $\sin n$ is a very strange number. (It's certainly not something we can easily write down in terms of numbers familiar to us.) But notice that $0 \leq (\sin n)^2 \leq 1$, always. Therefore, we can do a direct comparison test with

$$a_n = \frac{(\sin n)^2}{n^3}, b_n = \frac{1}{n^3}.$$

  Since $\sum b_n$ converges, being a p-series with $p = 3$, the direct comparison test tells us that $\sum a_n$ converges as well. (Forget about trying to calculate what the exact value of this series is, that's either very hard or currently impossible.)
- Strictly speaking, to apply the direct comparison test we only need to know that $0 \leq a_n \leq b_n$ for all n past a certain point, not all $n \geq 1$. For example, consider the series $\sum_{n=1}^{\infty} \frac{n+12}{n^2 4^n}$. Notice that $n + 12 \leq n^2$ when $n \geq 4$. Then we can apply direct comparison to the series with

$$a_n = \frac{n+12}{n^2 4^n}, b_n = \frac{1}{4^n},$$

  even though $a_n \leq b_n$ is not true for $n = 1, 2, 3$. Again, we can ignore these terms because the convergence and divergence of $\sum a_n, \sum b_n$ is not impacted by a finite number of terms; in this case, the first four terms of the series. In any case, notice that $\sum b_n = \sum \frac{1}{4^n}$, and this series converges since it is a geometric series with ratio $r = 1/4$.
- Consider the series $\sum_{n=1}^{\infty} 1/n!$. We will learn an easier way to check the convergence or divergence of this series in the near future, but for now let's try to use a comparison test.

  If we try $a_n = 1/n!, b_n = 1/n$, then it is certainly true that $a_n \leq b_n$ for all n. However, notice that $\sum b_n$ diverges, so with this setup the direct comparison test cannot tell us anything. However, notice that if we let $a_n = 1/n!, b_n = 1/n^2$, we might be able to use the direct comparison test. The first few terms of these two sequences are $1/1, 1/2, 1/6, 1/24, 1/120, \ldots$, vs $1/1, 1/4, 1/9, 1/16, 1/25, \ldots$. So while $a_n \leq b_n$ is not true for $n = 1, 2, 3$, it certainly seems to be true when $n \geq 4$. Let's check that this is indeed the case. We want to show that $a_n/b_n \leq 1$ for $n \geq 4$. Let's write out a_n/b_n:

$$\frac{a_n}{b_n} = \frac{1/n!}{1/n^2} = \frac{n^2}{n!} = \frac{n \cdot n}{n \cdot (n-1) \cdot (n-2) \cdots (1)}.$$

Since $n/n = 1$, and $n/(n-1) \leq 2$ for all n, this tells us that

$$\frac{a_n}{b_n} \leq \frac{2}{(n-2)!},$$

at least when $n \geq 2$. The right hand side of this expression is evidently ≤ 1 when $n \geq 4$, which shows what we wanted to show. So we can apply the direct comparison test to $a_n = 1/n!, b_n = 1/n^2$, and since $\sum b_n$ converges, $\sum a_n$ does as well.

2. The Limit Comparison test

There is a test very closely related to the direct comparison test, known as the limit comparison test. Suppose we have two series, $\sum a_n, \sum b_n$, with $0 < a_n, b_n$, and we know that $\lim_{n\to\infty} a_n/b_n = c$, where c is some positive (in particular, nonzero) number. This means that a_n is 'more or less' equal to b_n times c, so we should expect $\sum a_n$ to converge exactly when $\sum b_n$ converges. As a matter of fact, it's not too difficult to make this more precise to prove the following test:

The limit comparison test. Let $\sum a_n, \sum b_n$ be series with $0 < a_n, b_n$ for all n such that $\lim_{n\to\infty} a_n/b_n = c$, where c is a positive number. Then $\sum a_n$ converges exactly when $\sum b_n$ converges.

In principle it should be possible to just use the direct comparison test whenever you want to use the limit comparison test, but in certain situations the limit comparison test is easier to use.

Examples.

- Determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+\sqrt{n}}{\sqrt{n^3+7n^2}}$.

  This series looks a little crazy, and it is not so clear exactly how or if we should use direct comparison on it. Fortunately, this is the type of series well suited to limit comparison. Notice that the dominant term in the numerator, as n gets large, is n. In the denominator, the dominant term under the square root sign is n^3. So perhaps we can try using limit comparison against the series $\sum \frac{n}{n^{3/2}}$. Let's see what happens, with

  $$a_n = \frac{n+\sqrt{n}}{\sqrt{n^3+7n^2}}, b_n = \frac{n}{n^{3/2}} = \frac{1}{n^{1/2}}.$$

  We want to take the limit, as $n \to \infty$, of the fraction

  $$\frac{a_n}{b_n} = \frac{n+\sqrt{n}}{\sqrt{n^3+7n^2}} \cdot \frac{1/n}{1/n^{3/2}}.$$

  The reason for writing $1/b_n$ in this funny way is because we are going to multiply the numerators together, separately from the denominators:

  $$\frac{n+\sqrt{n}}{\sqrt{n^3+7n^2}} \cdot \frac{1/n}{1/n^{3/2}} = \frac{1+1/\sqrt{n}}{\sqrt{1+7/n}}.$$

  With this expression, it is easy to see that as $n \to \infty$, the limit of this is equal to 1. So we can apply the limit comparison theorem. Since $\sum b_n$ is a p-series with

$p = 1/2 < 1$, $\sum b_n$ diverges, so the limit comparison theorem says $\sum a_n$ diverges as well.

- Determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{(2n^2 - 3n + 4)5^n}$.

  There are a variety of ways to solve this problem. Let's use limit comparison. If a_n is the nth term of the given series, let's choose $b_n = 1/5^n$. Then

  $$\frac{a_n}{b_n} = \frac{3n^2}{(2n^2 - 3n + 4)5^n} \cdot 5^n = \frac{3n^2}{(2n^2 - 3n + 4)}.$$

  This expression tends to $3/2$ as $n \to \infty$, so we can use the limit comparison test. And since $\sum b_n$ converges (being a geometric series with ratio $r = 1/5$), we can conclude that the original series a_n converges as well.

Knowing when to use the comparison tests takes a little bit of practice. In general, if you have a series which 'looks like' a simpler series whose convergence or divergence you can determine using some other test (p-series, geometric, integral), then you might be able to use a comparison test if you can actually show that it is applicable. The limit comparison test is usually handy if you see polynomials in n in the numerator and denominator of a series. The direct comparison test can be useful if you see expressions where there is some variation in the numerator or denominator, but the variation is bounded; for example, something like adding $(-1)^n$ or $\sin n$, which is always of absolute value ≤ 1.

Ultimately, the best way to learn how and when to use the comparison tests is to work out many different problems. Your pattern recognition abilities will improve with practice.