MATH 8 CLASS 25 NOTES, 11/17/2010

Now that we have some idea of how to define a limit for multivariable functions, and how the behavior of limits of multivariable functions can be more complicated than those of single-variable functions, we begin to study how we should define derivatives of multivariable functions.

1. Partial Derivatives

Recall that the definition of the derivative of a single variable function $f(x)$ is the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Notice that there is no obvious way to extend this definition to functions of several variables. Instead, we will compromise our search for the proper definition of a derivative right now and content ourselves by defining a 'restricted' type of derivative.

Given a function $f(x, y)$ of two variables, we might consider what happens if we freeze one of the variables x, y, and then think of the resulting function as a function of the remaining variable. For example, if we freeze $x = 1$, then we can think of $f(1, y)$ as a function of y and then take the derivative of that function in the usual way. We call such a derivative a partial derivative of $f(x, y)$ with respect to the variable y.

More generally, we define partial derivatives f_x, f_y by the limits

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

In each definition, we freeze one of the variables (y for f_x, x for f_y) and use the usual definition of derivative for the remaining variable. In practice, taking derivatives is just as easy as taking derivatives of functions of a single variable.

Examples.

- Find the partial derivatives of $f(x, y) = x^2y + y^3$. To find f_x, we treat y as a constant and then take the derivative of $f(x, y)$ with respect to the variable x. We find $f_x(x, y) = 2xy$. Similarly, $f_y(x, y) = x^2 + 3y^2$.
- Find the partial derivatives of $f(x, y) = \ln(xy^2)$. Again,

$$f_x(x, y) = \frac{1}{xy^2}y^2, f_y(x, y) = \frac{1}{xy^2}2xy.$$

- Let $f(x, y) = xy + y^2$. Find the tangent lines to the curves obtained by slicing $f(x, y)$ using the planes $x = 1, y = 2$ at $(1, 2)$. When we slice $f(x, y)$ at $(1, 2)$ with $x = 1$, what we are doing is finding the tangent line to the curve $f(1, y)$ at $(1, 2)$. We see that $f_y(x, y) = x + 2y$, so $f_y(1, 2) = 5$. This line passes through $(1, 2, 6)$, and has direction vector given by $\langle 0, 1, 5 \rangle$. Indeed, we see that the x-coordinate of this line never changes, which explains why the first coordinate equals 0. This means that the tangent line in question has equation $x = 1, y = 2 + t, z = 6 + 5t$.

  Similarly, we can find the equation for the tangent line when we fix $y = 2$ by looking at $f_x(x, y) = y$. In this case, $f_x(1, 2) = 2$, so the tangent line in question passes through $(1, 2, 6)$ and has slope $\langle 1, 0, 2 \rangle$. Therefore, the tangent line in question has equation $x = 1 + t, y = 2, z = 6 + 2t$.

Recall that we had a geometric interpretation of $f'(x)$ as the slope of the tangent line to $f(x)$. Similarly, a partial derivative like $f_x(a,b)$ has a geometric interpretation as the slope of the tangent line to the curve obtained by slicing the graph of $f(x,y)$ by the plane $y = b$, at the point (a,b). Another way of saying this is that f_x is the rate of change of $f(x,y)$ when we hold y constant. Notice that this interpretation of a partial derivative makes it clear that $f_x(a,b)$ only depends on the values of $f(x,y)$ for $y = b$.

An alternative notation for partial derivatives is

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$$

for f_x, f_y. Another possible notation for f_x is $\partial_x f$.

Finally, we remark that the definition of partial derivatives holds not only for functions of two variables, but for functions in any number of variables. For example, if we have a function $f(x_1,\ldots,x_n)$, then its ith partial derivative is the limit

$$f_{x_i}(x_1,\ldots,x_n) = \lim_{h\to 0} \frac{f(x_1,\ldots,x_i+h,\ldots,x_n) - f(x_1,\ldots,x_n)}{h}.$$

We keep all variables except the ith fixed in the above limit. Therefore, when we actually calculate the partial derivatives of such a function, we treat x_i as a variable and every other x_j as a constant. The case of three-variable functions will be fairly common in Math 13.

Example. Calculate the three partial derivatives of $f(x,y,z) = \sin(xy)+ye^z$. We calculate $f_x(x,y,z) = y\cos xy$, $f_y(x,y,z) = x\cos(xy)+e^z$, and $f_z(x,y,z) = ye^z$.

2. Higher partial derivatives

Just like we can take multiple derivatives of a single-variable function $f(x)$ to obtain $f'(x), f''(x), f'''(x)$, etc., we can also take partial derivatives of partial derivatives to obtain higher partial derivatives. When we take partial derivatives, though, we need to pay attention to the order in which we take partial derivatives. For example, it is not a priori clear that f_{xy}, which is the function obtained by taking the partial derivative of f with respect to x first, and then the partial derivative of f_x with respect to y next, is the same as f_{yx}. We call f_{xy}, f_{yx} the mixed second partial derivatives of f.

Example. Calculate all the second partial derivatives of $f(x,y) = x^2y+e^{xy}$. We first calculate the two first partial derivatives:

$$f_x = 2xy + ye^{xy}, f_y = x^2 + xe^{xy}.$$

From this we can calculate the four second partial derivatives:

$$f_{xx} = 2y + y^2e^{xy}, f_{xy} = 2x + e^{xy} + xye^{xy}, f_{yx} = 2x + e^{xy} + xye^{xy}, f_{yy} = x^2e^{xy}.$$

Notice that f_{xy}, f_{yx} are equal. As a matter of fact, this will be true for almost every type of function we encounter, thanks to the following theorem:

Theorem. (Clairaut's Theorem) If $f(x,y)$ is a function such that f_{xy}, f_{yx} are both continuous at (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$.

That is, as long as f_{xy}, f_{yx} are both continuous functions, then they will also be equal to each other.

3. Tangent planes

Recall that the derivative of a single variable function can be interpreted as the slope of the tangent line to the graph of the function. We seek an analogous interpretation of the partial derivatives for multivariable functions.

The first and immediate difficulty which presents itself is the fact that there is no distinguished tangent line to a surface. For example, we can cut slices of the surface with various planes (such as when we hold x or y constant, when we take partial derivatives), and then find tangent lines to each curve which appears in a slice. However, no one of these lines by itself could stand in as a linear approximation to the surface, as a tangent line does for a curve.

As a matter of fact, if we are looking for a linear approximation to a surface, we should be looking for a 'tangent plane' instead of a tangent line. Since the graph of a function of two variables is two-dimensional, the linear approximation should also be two dimensional. Let us now think about how we would find a tangent plane.

This tangent plane should evidently contain every tangent line which we can obtain via the slicing procedure described above. Although it is not evident right now that the collection of these tangent lines should form a plane, we will prove this fact later. In any case, if the tangent plane does contain every tangent line as described above, it certainly should contain the tangent lines corresponding to the partial derivatives f_x, f_y.

Suppose we want to find the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0, z_0); of course, $z_0 = f(x_0, y_0)$. Then the two tangent lines corresponding to f_x, f_y have direction vectors $\langle 1, 0, f_x(x_0, y_0)\rangle, \langle 0, 1, f_y(x_0, y_0)\rangle$ respectively. To describe the tangent plane, we need to know a point on the plane, which we do in this case, as well as a normal vector. We have two vectors which lie on the plane; namely, the direction vectors we have found above, and they are not scalar multiples of each other, so their cross product will be a normal vector. We find that this cross product equals

$$\langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle$$

so the tangent plane will have equation

$$-f_x(x_0, y_0)x - f_y(x_0, y_0)y + z = -f_x(x_0, y_0)x_0 - f_y(x_0, y_0)y_0 + z_0.$$

We can rearrange this to the form

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

which formally looks like the point-slope form for the equation of a line.

Example. Find the tangent plane to $f(x, y) = xy + y^2$ at $(1, 2)$. This is the same example where we calculated the tangent lines for the slices of this function obtained by holding x or y constant. We find $f_x = y, f_y = x + 2y$, so $f_x(1, 2) = 2, f_y(1, 2) = 5$. The equation for the tangent line is thus

$$z - 6 = 2(x - 1) + 5(y - 2).$$

One can check, for instance, that the two tangent lines we calculated earlier both line on this plane.