MATH 8 CLASS 20 NOTES, 11/05/2010

Over the past two weeks we have developed the basic language of vectors and geometry in dimensions greater than 2. We carefully considered lines and planes in $\mathbb{R}^3$. These are both linear objects, which are already quite common, but to do calculus we want to consider more general, nonlinear functions. We begin by examining vector-valued functions, which are functions $\vec{r} : \mathbb{R} \to \mathbb{R}^n$, or more generally functions from some subset of the real numbers to $\mathbb{R}^n$. We will pay special attention to the cases $n = 2, 3$, where we obtain plane curves and space curves.

1. Vector-valued functions

A vector-valued function is simply any function $\vec{r}$ from some subset of the real numbers to $\mathbb{R}^n$. We often use the letter t for values in the domain of $\vec{r}$, and think of $\vec{r}(t)$ as giving a parametric description of f. Since $\vec{r}(t)$ is in $\mathbb{R}^n$, we can think of $\vec{r}(t)$ as a vector; hence the name vector-valued function.

Notice that we can also write $\vec{r}(t) = \langle r_1(t), r_2(t), \ldots, r_n(t)\rangle$, where $r_1(t), \ldots, r_n(t)$ are scalar valued functions (functions from $\mathbb{R}$ to $\mathbb{R}$). If we are studying vector-valued functions in three dimensions, we may write $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$. Each of these $r_i(t)$ is called a component function of $\vec{r}(t)$. When we describe vector-valued functions we often do so by providing the component functions of the vector-valued function.

Examples.

- Let $\vec{r}(t) = \langle \cos t, \sin t\rangle$. Then $\vec{r}(t)$ describes the unit circle in $\mathbb{R}^2$, and has period 2π.
- Let $\vec{r}(t) = \langle 1/(t-3), \sqrt{t-1}, e^t\rangle$. What is the domain of $\vec{r}$? The domain is the set of points for which the function is defined; in order for this function to be defined, every one of its component functions need to be defined. $1/(t-3)$ is defined everywhere except $t = 3$, while $\sqrt{t-1}$ is exactly when $t \geq 1$, and e^t is defined everywhere. Therefore, the domain of $\vec{r}$ is the set of all numbers $t \geq 1$, except $t = 3$. (Sketching $\vec{r}$ is probably much too difficult to do by hand.)
- Let $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$. What does the graph of $\vec{r}$ look like? Notice that the x, y coordinates describe a circle. The z coordinate is constantly increasing; therefore the graph of $\vec{r}$ looks like a vertical helix. Let $\vec{s}(t) = \langle \cos t, -\sin t, -t\rangle$. Notice that $\vec{r}$ and $\vec{s}$ trace out the exact same graph, because $\vec{r}(-t) = \vec{s}(t)$. Despite the fact that $\vec{r}, \vec{s}$ have the same graph, they are still different functions. In general, it is possible for two different vector-valued functions to trace out the exact same curve.

If we want to do calculus, we should have an idea of what a limit is. Fortunately, the limit of a vector-valued function is easily defined in terms of the limits of its component functions. If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is a vector-valued function, and c some real number, then the limit of $\vec{r}(t)$ as $t \to c$ is the vector

$$\lim_{t\to c} \vec{r}(t) = \langle \lim_{t\to c} f(t), \lim_{t\to c} g(t), \lim_{t\to c} h(t)\rangle$$

assuming that each of the limits of the component functions also exists. If any one of the limits of $f(t), g(t), h(t)$ do not exist, then the entire limit of $\vec{r}(t)$ also does not exist. In practice, evaluating limits of vector-valued functions simply consists of evaluating several familiar limits of scalar-valued functions.

Example. Let

$$\vec{r}(t) = \left\langle \frac{\sin t}{t}, \frac{t^2+2t}{t^2+t}, e^t \right\rangle.$$

Find $\lim_{t\to 0} \vec{r}(t)$. To solve this problem we simply take the limits of each of the three component functions as $t \to 0$. The first is equal to 1, the second equal to 2, and the last equal to 1. Therefore, $\lim_{t\to 0} \vec{r}(t) = \langle 1, 2, 1\rangle$.

We can also sometimes ask slightly more elaborate questions about intersections of various surfaces with each other or with other curves.

Examples.

- Give a vector-valued function whose graph describes the intersection between the cone $x^2+y^2 = z^2, z \geq 0$ and the sphere $x^2+y^2+z^2 = 2$. (You do not necessarily need to know that $x^2+y^2 = z^2$ actually gives a cone yet, since we will cover surfaces in $\mathbb{R}^3$ later.) If we draw a sketch of these two objects, it is fairly clear that their intersection should be a circle. What we are looking for is a function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ such that $\langle f(t), g(t), h(t)\rangle$ describes every point in the intersection of the cone and the sphere. In particular, we need $f^2 + g^2 = h^2$, as well as $f^2 + g^2 + h^2 = 2$. From this, it is immediately clear that $h(t) = 1$ (recall that $h(t) \geq 0$). We thus need to find f, g such that $f^2 + g^2 = 1$. However, notice that this has the form of an equation for a circle, and so we can take $f(t) = \cos t, g(t) = \sin t$. In summary, $\vec{r}(t) = \langle \cos t, \sin t, 1\rangle$ is a vector-valued function which describes this intersection.
- Find all points of intersection between the curve given by $\vec{r}(t) = \langle t, t^2, 2t^3\rangle$ and the cylinder with equation $x^2 + y^2 = 2$. To solve this problem, we simply plug in the component functions for $\vec{r}$ into the corresponding coordinates in the equation for the cylinder, since we are looking for points on the curve which also lie on the cylinder. This gives the equation $t^2+t^4 = 2$, or $t^4+t^2-2 = 0$. We can factor this as $(t^2+2)(t^2-1) = 0$, so the real solutions to this equation are given by $t = -1, 1$. To find the actual coordinates of the points of intersection, substitute these values of t into the parametric equation which describes $\vec{r}(t)$: this yields the points $(-1, 1, -2)$ and $(1, 1, 2)$.

2. Differentiating and integrating vector-valued functions

Now that we know how to take limits of vector-valued functions, and have seen that taking limits of vector-valued functions is not any harder than taking limits of scalar-valued functions, let us define derivatives and integrals of vector-valued functions.

The derivative of a vector-valued function is defined in the same way as the derivative of a scalar-valued function: using the limit of a difference quotient. More specifically, if $\vec{r}(t)$ is a vector-valued function, then its derivative at t, written $\vec{r}'(t)$, is defined to be the limit

$$\vec{r}'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

if this limit exists. Notice that this is a limit of a vector-valued function, so the result will still be a vector. Fortunately, we will basically never need to use this definition, because it is easy to show that if $\vec{r}(t) = \langle r_1(t), \ldots, r_n(t)\rangle$, then $\vec{r}'(t)$ exists if and only if each of the derivatives $r_1'(t), \ldots, r_n'(t)$ exists, and then $\vec{r}'(t) = \langle r_1'(t), \ldots, r_n'(t)\rangle$. In other words, taking derivatives of vector-valued functions reduces to taking the derivatives of several scalar-valued functions, just like when we take limits!

We can interpret the vector $\vec{r}'(t)$ as the tangent vector to $\vec{r}$ at t. This vector is a direction vector for the tangent line for $\vec{r}$ which passes through $\vec{r}(t)$. We will often have occasion to use the unit tangent vector for $\vec{r}$ at t, which is defined to be the vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}.$$

This is evidently the unique unit vector which points in the same way as $\vec{r}'(t)$.

Examples.

- Let $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$ describe a vertical helix. What is $\vec{r}'(t)$ and $\vec{T}(t)$? What is the equation for the tangent line to $\vec{r}(t)$ at $t = \pi/2$?

  Using the various definitions above, we see that $\vec{r}'(t) = \langle -\sin t, \cos t, 1\rangle$. Since $|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1^2} = \sqrt{2}$, the unit vector $\vec{T}(t) = 1/\sqrt{2}\vec{r}'(t)$. Finally, to determine the equation for the tangent line to $\vec{r}(t)$ at $t = \pi/2$, we find $\vec{r}'(\pi/2) = \langle -1, 0, 1\rangle$. This is a direction vector for the tangent line, which also passes through $\vec{r}(\pi/2) = \langle 0, 1, \pi/2\rangle$. Therefore, the tangent line in question is given parametrically by

$$x = -t, y = 1, z = \pi/2 + t.$$

- Let $\vec{r}(t) = \langle e^t - 1, \ln(t^2 + e), \sin t\rangle$. Determine $\vec{r}'(t)$, and find the tangent line to $\vec{r}$ at $t = 0$.

  Again, we find that

$$\vec{r}'(t) = \left\langle e^t, \frac{2t}{t^2 + e}, \cos t \right\rangle.$$

  To find the tangent line to $\vec{r}$ at $t = 0$, we first find $\vec{r}'(0) = \langle 1, 0, 1\rangle$. Also, $\vec{r}(0) = \langle 0, 1, 0\rangle$, so the tangent line has equation

$$x = t, y = 1, z = t.$$

The derivatives of vector-valued functions satisfy a variety of properties analogous to those satisfied by scalar-valued functions. Of particular note are those involving the various types of 'products' we can form with vector-valued functions. Suppose $\vec{r}, \vec{s}$ are vector valued functions, and $u(t)$ a scalar valued function. Then we have the following properties:

$$\begin{aligned}
r \vec{+} s'(t) &= \vec{r}'(t) + \vec{s}'(t) \\
(u(t)\vec{r}(t))' &= u'(t)\vec{r}(t) + u(t)\vec{r}'(t) \\
(\vec{r}(t) \cdot \vec{s}(t))' &= \vec{r}'(t) \cdot \vec{s}(t) + \vec{r}(t) \cdot \vec{s}'(t) \\
(\vec{r}(t) \times \vec{s}(t))' &= \vec{r}'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}'(t) \\
\vec{r}(u(t))' &= \vec{r}'(u(t))u'(t)
\end{aligned}$$

The first identity is straightforward. The middle three are all analogues of the product rule; even though scalar multiplication, dot product, and cross product are all defined quite differently, they all formally obey a rule similar to the product rule for scalar-valued functions. The reason this is true is because we can ultimately reduce derivatives involving scalar multiplication, dot product, and cross product into derivatives involving products of various component functions. Finally, the last rule is the analogue of the chain rule.

Example. As an interesting application of one of the above rules, let us show that if an object moves in a circle, its velocity vector is always orthogonal to its position vector. This is a concept that frequently appears in classical physics. Let $\vec{r}(t)$ describe the motion of an object; we can set the center of the circle the object moves in equal to the origin. Then $\vec{r}(t)$ has constant length. In particular, this means that $\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2$ is constant, so its derivative is equal to 0. On the other hand,

$$(\vec{r}(t) \cdot \vec{r}(t))' = 2\vec{r} \cdot \vec{r}' = 0,$$

which implies that $\vec{r} \cdot \vec{r}' = 0$, or that $\vec{r}, \vec{r}'$ are orthogonal, as desired.