MATH 8 CLASS 2 NOTES, 9/24/2010

1. Series: An Introduction

We will spend our time studying sequences of a special form. A series is an expression of the form

$$\sum_{n=1}^{\infty} a_n$$

where the a_n are numbers (a sequence of numbers). A series does not necessarily have to start at the index $n = 1$; for example, a series could start at $n = 0$, $n = 2$, or even $n = -1$ or some other negative number. The individual members a_n are called the *terms* of the series.

What exactly does the above notation mean? It is evidently a generalization of the summation notation

$$\sum_{n=1}^{N} a_n = a_1 + a_2 + \ldots + a_N$$

with the N replaced by ∞. This suggests that if we want to interpret the value of a series $\sum_{n=1}^{\infty} a_n$, we should study the sequence of partial sums

$$s_N = \sum_{n=1}^{N} a_n = a_1 + \ldots + a_N$$

These partial sums form a sequence $\{s_n\}$ which either converges or diverges. If $\{s_n\}$ converges to the limit L, then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to (or has *sum* or *value*) L, and if $\{s_n\}$ diverges, then we say the associated series diverges.

Examples.

- Consider the series $\sum_{n=1}^{\infty} 1$. Does this series converge or diverge, and if it converges, to what limit? To determine if this series converges or diverges, we examine the partial sums. In this case, we can easily see that $s_n = n$, and so the sequence of partial sums diverges to infinity. Therefore, the series $\sum_{n=1}^{\infty} 1$ diverges.
- Does the series $\sum_{n=1}^{\infty}(-1)^n$ converge or diverge, and if it converges, to what limit? The partial sums are $s_1 = -1, s_2 = 0, s_3 = -1, s_4 = 0, \ldots$. Therefore, this series also diverges, since its partial sums oscillate between -1 and 0.
- Does the series $\sum_{n=1}^{\infty} 1/2^n$ converge or diverge, and if it converges, to what limit? This series is $1/2 + 1/4 + 1/8 + \ldots$. Notice that $s_n = 1/2 + \ldots + 1/2^n$. We can rewrite this sum as follows:

$$\frac{1}{2} + \ldots + \frac{1}{2^n} = \frac{(1/2)(1 - 1/2^n)}{1 - (1/2)}$$

If this last expression looks complicated, notice that the first few partial sums are $s_1 = 1/2, s_2 = 3/4, s_3 = 7/8, \ldots$. So a general formula for s_n looks an awful lot like $s_n = 1 - 1/2^n$. You can check on your own that this is equal to the more complicated looking expression above. In any case, as $n \to \infty$, $s_n \to 1$. Therefore, the series above converges to 1.

The last example is very special. In general, we can rewrite the expression $1 + r + r^2 + \ldots + r^n$ (if $r \neq 1$) as

$$1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Therefore, a series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots$$

has partial sums

$$s_n = \frac{a(1 - r^{n+1})}{1 - r}$$

This expression allows us to easily determine the limit of the partial sums, as $n \to \infty$. If $|r| < 1$, then $r^{n+1} \to 0$ as $n \to \infty$, and the series converges to

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}$$

On the other hand, if $|r| > 1$, then r^{n+1} diverges, and the partial sums s_n diverge as well. If $|r| = 1$, we cannot factor the partial sums as above, but then the series in question just becomes $a + a + \ldots$, which obviously converges if and only if $a = 0$, or the series $a - a + a - a + \ldots$, which also only converges if $a = 0$. (Notice that if $a = 0$, then the series just becomes $0 + 0 + 0 + \ldots$, which converges to 0).

A series of the type

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots$$

where $a \neq 0$ is called a <u>geometric series</u>, and we have just shown that a geometric series either has sum $a/(1 - r)$, if $|r| < 1$, or diverges, if $|r| \geq 1$. The number a is called the <u>initial term</u> of the geometric series, and the number r is called the <u>ratio</u>.

Examples.

- We have already seen that the series $1/2 + 1/4 + 1/8 + \ldots = 1$. This is a geometric series with initial term $a = 1/2$ and ratio $r = 1/2$, and converges to $a/(1 - r) = 1/2/(1 - 1/2) = 1$.
- Consider the series $\sum_{n=1}^{\infty} (-3)^n/4^{n+1}$. This is also a geometric series, with initial term $a = -3/16$ (notice that the series starts at index $n = 1$) and ratio $r = -3/4$. Since $|r| < 1$, this series converges, and converges to the limit $a/(1 - r) = -3/16/(7/4) = -3/28$.
- Consider the series $\sum_{n=2}^{\infty} 2^{2n}/3^{n+2}$. This is a geometric series, with initial term $a = 16/81$ and ratio $r = 4/3$. Since $|r| > 1$, the series diverges.

As these examples show, if we can identify a series as a geometric series, it is easy to test for convergence or divergence of the series, and to calculate the sum of the series if it converges. Geometric series are important, but they are hardly the only type of series that we will encounter. For other types of series, we need to develop a set of tools which will allow us to determine the convergence or divergence of those series. However, the question of determining the sum of a convergent series is, in general, very difficult, and geometric series are one of the few types of series where the sum can be easily determined.

Sometimes we might get lucky and be able to calculate the value of a convergent series even if it is not geometric. More specifically, suppose we are fortunate enough to be given a series whose partial sums can be simplified to a form where there are only a few terms, each easily analyzed as $n \to \infty$. Then we might be in a situation to determine the value of that series. This is best illustrated by an example.

Example. (Example 12.2.6 from the book) Determine whether the series $\sum_{n=1}^{\infty} 1/(n^2+n)$ converges or diverges, and if it converges, determine the limit. A good place to start when thinking about convergence or divergence is to first check if the series is geometric. A bit of thought shows that it is not (for example, the first few terms in the sum are $1/2, 1/6, 1/12, \ldots$, and this already shows that the series is not geometric). Right now, we don't have any other tools for determining convergence or divergence, so let's calculate the first few partial sums: $s_1 = 1/2, s_2 = 1/2+1/6 = 2/3, s_3 = 2/3+1/12 = 3/4, \ldots$. So there seems to be a pattern, where $s_N = N/(N+1)$. However, we should find a way to show that this is correct for all n, not just the first few n we are able to test.

If you are clever, or have learned about partial fraction decomposition, you might realize that

$$a_n = \frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

With this expression for the general term of the series in hand, let's write out what the Nth partial sum looks like. We have

$$s_N = a_1 + \ldots + a_N = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{N} - \frac{1}{N+1}\right).$$

Notice that something very convenient happens! All the terms in the above sum cancel except the very first and the very last. So we find that $s_N = 1 - 1/(N+1) = N/(N+1)$, as desired. With this simple expression for s_N in hand, it is obvious that $s_N \to 1$ as $N \to \infty$. So the above series converges to 1. This is an example of what is known as a 'telescoping' series, where lots of cancellation occurs in a partial sum and what's left is a simple expression which we can easily analyze as $N \to \infty$. In general, telescoping sums are rare, and it is not always easy to determine whether a given sum will be telescoping or not.

2. The nth term test for divergence

We will begin our search for 'tests for convergence/divergence' at a fairly simple level. Suppose we have a series

$$\sum_{n=1}^{\infty} a_n$$

where the sequence $\{a_n\}$ converges to a nonzero limit L. For example, think of the constant sequence given by $a_n = L$. In this specific example, the partial sums of this series are $s_n = nL$, which is evidently unbounded (since $L \neq 0$), so the series diverges. It is not too difficult to transfer this argument over to any sequence $\{a_n\}$ with nonzero limit L. (Here is a sketch for $L > 0$. For sufficiently large n, a_n is very close to L – close enough to ensure that $a_n > L/2$. Then the partial sums $s_n \geq S + nL/2$, for some number S, and as $n \to \infty$, $s_n \to \infty$ as well. Change the signs in the argument for $L < 0$.) It is slightly more difficult to generalize this argument to a series given by a divergent sequence $\{a_n\}$, but it is possible. This gives the following test for divergence:

Theorem. (The nth term test for divergence.) Suppose the sequence $\{a_n\}$ does not converge to 0. Then the series $\sum_{n=1}^{\infty} a_n$ diverges.

A few remarks about this test are in order. This test is logically equivalent to the statement "If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$". However, it is certainly not always true that if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. Therefore, you can NEVER use this test to claim that a series converges. To show that a series converges, you need to use some other test, and right now, the only general test we have available for us is testing whether a geometric series converges or diverges.

When you use this test on your written homework or exams, make sure to actually say that you are using the test, and show that the sequence $\{a_n\}$ does not converge to 0. Do not just say that a series diverges without telling us what you are using and checking that you are allowed to use the test you are using. Justification is actually more important than getting the correct answer, since you can always get the correct answer by guessing (half chance if asking about convergence or divergence), but you can only get the correct justification if you know how to solve the problem.

Examples.

- Consider $\sum_{n=1}^{\infty} n/(2n+1)$. Does this series converge or diverge? Notice that the limit of $n/(2n+1)$ as $n \to \infty$ is $1/2$. Since this is nonzero, the series diverges, by the nth term test for divergence.
- Consider $\sum_{n=1}^{\infty} 2^n/n^2$. Does this series converge or diverge? Notice that $2^n/n^2$ diverges to infinity, so the series also diverges by the nth term test for divergence.
- Consider $\sum_{n=1}^{\infty} (-1)^n/n$. Does this series converge or diverge? Well, $\lim_{n\to\infty} (-1)^n/n = 0$. Therefore, we cannot conclude that this test converges or diverges – we do not know enough yet to check whether this series converges or diverges. (We will see that this series converges in a few days.)

Here's a quick summary of what we covered today:

- A geometric series is a series of the form $\sum_{n=0}^{\infty} ar^n$, for some constants $a \neq 0, r$. Then this series converges if $|r| < 1$ and diverges if $|r|/geq1$. If the series converges, the value of the series is $a/(1-r)$.
- If you are lucky, a series might be a telescoping series, which means that when writing out the partial sums, lots of cancellation occurs which leaves an expression which you can easily analyze. Sometimes you might have to do a little bit of algebraic manipulation to easily see that a series is telescoping.
- The nth term test for divergence says that the series $\sum a_n$ diverges if the sequence $\{a_n\}$ does not converge to 0. You cannot use this test to check that a series converges – only that it diverges. Furthermore, there are series $\sum a_n$ with $\lim a_n = 0$ which still diverge. For example, the harmonic series $\sum 1/n$ is such a series.