MATH 8 CLASS 12 NOTES, 10/18/2010

Today, we will be interested in learning how to calculate integrals which involve various products of trigonometric functions. We will make extensive use of trigonometric identities to transform integrands to expressions where we can easily use u-substitution. Before embarking on the use of identities per se, let us recall the types of trigonometric integrals for which we can use u-substitution.

Examples.

- Calculate $\int \sin^3 x \cos x\, dx$. The substitution $u = \sin x, du = \cos x\, dx$ seems like the natural substitution to make here. This transforms the integral to $\int u^3\, du = u^4/4 + C = \sin^4 x/4 + C$.
- Calculate $\int \cos^5 x \sin x\, dx$. This example isn't much different from the previous example – make the substitution $u = \cos x, du = -\sin x\, dx$, which transforms the integral to $\int -u^5\, du = -u^6/6 + C = -\cos^6 x/6 + C$.

In both these examples, we could easily integrate a power of cos or sin as long as one term of the other trigonometric function was present in the integrand. However, how do we evaluate an integral like $\int \cos^5 x \sin^3 x\, dx$, where there may be high powers of both cos and sin? The answer lies in the fundamental trigonometric identity

$$\sin^2 x + \cos^2 x = 1$$

This is a very important identity, not just for evaluating trigonometric integrals, but in mathematics in general, and is nothing more than the fact that $(\cos x, \sin x)$ is a point on the unit circle, which therefore has distance 1 from the origin.

1. Odd powers of sin or cos

If we have either an odd power of sin or cos in an integral of the form $\int \sin^m x \cos^n x\, dx$, then we can eliminate all but one power of that trigonometric function by applying the identity $\sin^2 x + \cos^2 x = 1$. We'll calculate some examples:

Examples.

- Calculate $\int \sin^4 x \cos^3 x\, dx$. Transform all powers of cos except one into sins by using $\sin^2 x + \cos^2 x = 1$ in the following way: since $\cos^2 x = 1 - \sin^2 x$, replace $\cos^2 x$ in the expression above to obtain

$$\int \sin^4 x \cos^3 x\, dx = \int \sin^4 x(1 - \sin^2 x) \cos x\, dx$$

  After expanding the terms with sin, we obtain an integral we can evaluate using u-substitution, with $u = \sin x$:

$$\int \sin^4 x(1 - \sin^2 x) \cos x\, dx = \int u^4 - u^6\, du = \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$$

- Calculate $\int \sin^6 x \cos^5 x\, dx$. Just as in the previous example, we eliminate all but one copy of $\cos x$ by using $\cos^2 x = 1 - \sin^2 x$:

$$\int \sin^6 x \cos^5 x\,dx = \int \sin^6 x(1-\sin^2 x)^2 \cos x\,dx$$

Notice that we need to expand $(1-\sin^2 x)^2$ at some point. Nevertheless, we can make the u-substitution $u = \sin x, du = \cos x\,dx$ right now to obtain

$$\begin{aligned}\int \sin^6 x(1-\sin^2 x)^2\cos x\,dx &= \int u^6(1-u)^2\,du = \int u^6 - 2u^8 + u^{10}\,du \\ = \frac{u^7}{7} - \frac{2u^9}{9} + \frac{u^{11}}{11} + C &= \frac{\sin^7 x}{7} - \frac{2\sin^9 x}{9} + \frac{\sin^{11} x}{11} + C\end{aligned}$$

- If the power of sin is odd, we can replace every copy of $\sin x$ except one using $\sin^2 x = 1-\cos^2 x$. For example, consider the integral $\int \sin^3 x\cos^2 x\,dx$. Then this identity yields

$$\int \sin^3 x\cos^2 x\,dx = \int \sin x(1-\cos^2 x)\cos^2 x\,dx$$

The u-substitution $u = \cos x, du = -\sin x\,dx$ gives

$$\int \sin x(1-\cos^2 x)\cos^2 x\,dx = -\int u^2 - u^4\,du = -\frac{u^3}{3} + \frac{u^5}{5} + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.$$

- If both sin and cos have odd powers, then you can choose to eliminate all but one copy of either function. However, in practice, you probably want to eliminate all but one power of the term which has a lower power, to minimize the amount of polynomial expansion you have to do. For example, consider the integral $\int \sin^5 x\cos^3 x\,dx$. Then we might choose to eliminate all but one copy of $\cos x$:

$$\int \sin^5 x\cos^3 x\,dx = \int \sin^5 x(1-\sin^2 x)\cos x\,dx$$

Make the u-substitution $u = \sin x, du = \cos x\,dx$, to obtain

$$\int \sin^5 x(1-\sin^2 x)\cos x\,dx = \int u^5 - u^7\,du = \frac{u^6}{6} - \frac{u^8}{8} + C = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C.$$

- In the previous example, you might be wondering what would have happened had we instead chosen to eliminate all but one copy of sin. Then we would obtain some sort of final answer in terms of a polynomial in cos, which certainly does not look like the answer in the previous example. Can you think about why these two answers are actually equal? As a simpler example of this phenomenon, consider the integral $\int \sin^3 x\cos x\,dx$. The simple way to do this integral is just to let $u = \sin x$, which give us an answer of $\sin^4 x/4 + C$. However, suppose we make the substitution $\sin^2 x = 1-\cos^2 x$. Then this integral becomes

$$\int (1-\cos^2 x)\cos x\sin x\,dx = -\int (1-u^2)u\,du = -\frac{u^2}{2} + \frac{u^4}{4} + C == -\frac{\cos^2 x}{2} + \frac{\cos^4 x}{4} + C.$$

Even though this doesn't look like the simpler answer we got, closer inspection will show that they actually are equal. For example, if we take the difference of the answers, we obtain

$$\frac{\sin^4 x - \cos^4 x}{4} + \frac{\cos^2 x}{2} + C = \frac{(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)}{4} + \frac{\cos^2 x}{2} + C = \frac{\sin^2 x - \cos^2 x}{4} + \frac{\cos^2 x}{2} + C.$$

At this point, we use $\sin^2 x = 1 - \cos^2 x$ to convert everything to cos:

$$\frac{\sin^2 x - \cos^2 x}{4} + \frac{\cos^2 x}{2} + C = \frac{1 - 2\cos^2 x}{4} + \frac{\cos^2 x}{2} + C = \frac{1}{4} + C = C$$

Therefore, these two seemingly different answers really are equal!

2. Even powers of sin and cos

What happens if we have an integral with even powers of both cos and sin? For example, the seemingly simple integral $\int \sin^2 x\,dx$ falls under this category. Then a use of $\sin^2 x = 1 - \cos^2 x$ does no good, because we end up with other integrals we do not know how to evaluate. To solve integrals like this requires a new trigonometric identity. We can use the identity $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$ to obtain the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \cos^2 x = \frac{1 + \cos 2x}{2}, \sin x \cos x = \frac{\sin 2x}{2}$$

These identities provide the transformations we need to solve integrals with even powers of both sin and cos in them.

Examples.

- Evaluate $\int \sin^2 x\,dx$. We apply the identity above to transform this integral:

$$\int \sin^2 x\,dx = \int \frac{1 - \cos 2x}{2}\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

- Sometimes you need to use the above identities more than once. For example, consider the integral $\int \cos^4 x\,dx$. Then making the substitution with $\cos^2 x$ above yields

$$\int \cos^4 x\,dx = \int \frac{(1 + \cos 2x)^2}{4}\,dx = \int \frac{1 + 2\cos 2x + \cos^2 2x}{4}\,dx$$

At this point, we can integrate the first two terms in the sum in the last integrand, but to integrate $\cos^2 2x$ requires us to use the above identity again, with $2x$ in place of x. This yields

$$\int \frac{1 + 2\cos 2x + \cos^2 2x}{4}\,dx = \int \frac{1 + 2\cos 2x + \frac{1+\cos 4x}{2}}{4}\,dx = \frac{3}{8}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$$

where we skipped several simplification steps at the end. Repeated applications of identities like $\cos^2 x = (1 + \cos 2x)/2$ can lead to fairly messy expressions, so make sure to stay organized when solving problems like this.

3. Integrals with $\tan, \sec$

We now consider integrals which have various powers of tan and sec in them. The basic integration formulas we will use here are

$$\int \sec^2 x\,dx = \tan x + C, \int \tan x \sec x\,dx = \sec x + C$$

In addition, it is occasionally useful to know the not quite as elementary formulas

$$\int \tan x\,dx = \ln|\sec x| + C, \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

The basic trigonometric identity we will use here now is

$$\tan^2 x + 1 = \sec^2 x$$

This is easily obtained from $\sin^2 x + \cos^2 x = 1$ by dividing both sides by $\cos^2 x$.
Because the derivatives of $\tan x, \sec x$ are $\sec^2 x, \tan x \sec x$ respectively, the strategy behind using this identity will be to obtain integrals whose integrands are either $\tan^k x \sec^2 x$ or $\sec^k x(\sec x \tan x)$, because we can then apply the u-substitutions $u = \tan x, u = \sec x$ respectively.

Examples.

- Evaluate $\int \tan^3 x \sec^4 x\, dx$. If we see an even number of powers of sec, we should eliminate all but $\sec^2 x$ of those powers. In this example, we apply $\sec^2 x = 1+\tan^2 x$ to obtain
$$\int \tan^3 x \sec^4 x\, dx = \int \tan^3 x(1 + \tan^2 x) \sec^2 x\, dx$$
Make the u-substitution $u = \tan x, du = \sec^2 x\, dx$ to obtain
$$\int \tan^3 x(1 + \tan^2 x) \sec^2 x\, dx = \int u^3 + u^5\, du = \frac{u^4}{4} + \frac{u^6}{6} + C = \frac{tan^4 x}{4} + \frac{tan^6 x}{6} + C$$
- Evaluate $\int \sec^3 x \tan^3 x\, dx$. In this example, we have an odd number of tan in the integrand, and what we want to do is convert all of those except one copy of $\tan x$ into sec. In this case, we obtain
$$\int \sec^3 x \tan^3 x\, dx = \int \sec^3 x(\sec^2 x - 1) \tan x\, dx$$
At this point, we make the u-substitution $u = \sec x, du = \sec x \tan x\, dx$. One copy of the $\sec x$ needs to go into the du, so we obtain
$$\int \sec^3 x(\sec^2 x - 1) \tan x\, dx = \int u^4 - u^2\, du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$
- Odd powers of tan, by themselves, can be evaluated by repeatedly using $\tan^2 x = \sec^2 x - 1$. For example, consider $\int \tan^3 x\, dx$. One application of the above identity gives
$$\int \tan^3 x\, dx = \int (\sec^2 x - 1) \tan x\, dx = \frac{\tan^2 x}{2} - \ln|\sec x| + C$$
- Even powers of tan, by themselves, can also be handled similarly. For example, on the integral $\int \tan^6 x\, dx$, one application of $\tan^2 x = \sec^2 x - 1$ gives
$$\int \tan^6 x\, dx = \int \tan^4 x(\sec^2 x - 1)\, dx = \frac{\tan^5 x}{5} - \int \tan^4 x\, dx$$
We can evaluate $\int \tan^4 x\, dx$ by using this same method to reduce to an integral with $\int \tan^2 x\, dx = \int \sec^2 x - 1\, dx = \tan x - x$.
- Odd powers of $\sec x$ are substantially harder, although still solvable. For example, we can use the formula from exercise 50 of Chapter 8.1, which provides a 'reduction' formula for $\int \sec^n x\, dx$ by using integration by parts:

$$\int \sec^n x\,dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

We can repeatedly apply this formula until we obtain $\int \sec x\,dx = \ln|\sec x + \tan x| + C$. For example, this formula applied to $\int \sec^3 x\,dx$ gives

$$\int \sec^3 x\,dx = \frac{\tan x \sec x}{2} + \frac{1}{2}\int \sec x\,dx = \frac{\tan x \sec x}{2} + \frac{\ln|\sec x + \tan x|}{2} + C$$

Integrals with $\cot x, \csc x$ can be handled in a manner similar to integrals with $\tan x, \sec x$ by virtue of the identity $\cot^2 x + 1 = \csc^2 x$, and the fact that the derivatives of $\cot x, \csc x$ are $-\csc^2 x, -\cot x \csc x$, respectively.