MATH 8 CLASS 10 NOTES, 10/13/2010

Some notation from the concepts we covered in the last class: the partial sum of a Taylor series for a function $f(x)$ at the point $x = a$ up to the x^n term is sometimes called the nth Taylor polynomial for $f(x)$, and is written $T_n(x)$. Concretely, this means

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Let us look at more examples of Taylor series, and applications to other mathematical problems.

Examples.

- Calculate the Maclaurin series for $f(x) = \sin x$. To solve this problem, we need to be able to calculate derivatives of every order of f, evaluated at a. Fortunately, $f'(x) = \cos x, f''(x) = -\sin x, f^{(3)}(x) = -\cos x, f^{(4)}(x) = \sin x$, so these derivatives repeat, and we have $f(0) = 0, f'(0) = 1, f''(0) = 0, f^{(3)}(0) = -1, f^{(4)}(0) = 0, \ldots$. If we plug these values into the formula for Taylor series, we obtain

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^x x^{2n+1}}{(2n+1)!}$$

- Estimate $\sin 1$ to within an accuracy of $1/1000$. We use the Maclaurin series we found above:

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \ldots$$

  Notice that this is an alternating series! Therefore, we can use the error estimate for the alternating series test, instead of the Taylor remainder theorem, which is much more difficult to use. If we want to estimate $\sin 1$ to within 0.001 accuracy, we want the first discarded term to have size less than 0.001. Since $7! = 5040 > 1000, 1/7! < 0.001$, and therefore $1 - 1/3! + 1/5!$ estimates $\sin 1$ to within an error of 0.001.

  This example shows that there might be occasions where a Taylor series is also an alternating series. If that is the case, it is probably easier to apply an error estimate for the alternating series test, instead of using the more complicated Taylor's remainder theorem.

- Taylor series are also handy for evaluating certain limits. For example, suppose we want to calculate

$$\lim_{x\to\infty} \frac{\cos x - 1 + x^2/2}{2x^4}$$

  Replace $\cos x$ with its Maclaurin series, which is valid for all real x:

$$\lim_{x\to\infty} \frac{\cos x - 1 + x^2/2}{2x^4} = \frac{(1 - x^2/2 + x^4/4! - \ldots) - 1 + x^2/2}{2x^4} = \frac{x^4/4! - \ldots}{2x^4}$$

  All the $\ldots$ are higher order terms, and as $x \to \infty$ they vanish, so this limit is $1/48$. Alternatively, if we knew L'Hopital's rule, we could have also evaluated this limit.

- Calculate the following integral as a power series:
$$\int x^2 \arctan x^2 \, dx$$
For what values of x does your solution converge? For a problem like this, determine the power series expansion of the integrand, and then integrate that term-by-term. In this problem, we have
$$x^2 \arctan x^2 = x^2 \left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \dots \right) = x^4 - \frac{x^8}{3} + \frac{x^{12}}{5} - \frac{x^{16}}{7} + \dots .$$
If we integrate this term by term, we get
$$\int x^2 \arctan x^2 \, dx = \frac{x^5}{5} - \frac{x^9}{9 \cdot 3} + \frac{x^{13}}{13 \cdot 5} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(4n+5)(2n+1)} + C$$
This series is convergent for $|x| < 1$, as the ratio test will show. Also, at the endpoints $x = \pm 1$, one can show that this series converges there as well, by the alternating series test.
- Using the answer to the previous question, use the least number of terms possible to estimate the definite integral
$$\int_0^{1/2} x^2 \arctan x^2 \, dx$$
to an accuracy of 0.0001. If we plug in the bounds $0, 0.5$ to the answer in the previous question, we find that the definite integral in question is equal to the value of the alternating series
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n+5}(4n+5)(2n+1)} = \frac{1}{2^5 \cdot 5} - \frac{1}{2^9 \cdot 9 \cdot 3} + \frac{1}{2^{13} \cdot 13 \cdot 5} - \dots .$$
Since $0.0001 = 10^{-4}$, and $2^{13} \cdot 13 \cdot 5 > 10^4$, while $2^9 \cdot 9 \cdot 3 < 10^4$, this shows that
$$\frac{1}{2^5 \cdot 5} - \frac{1}{2^9 \cdot 9 \cdot 3}$$
estimates our integral to an accuracy of at least 0.0001.

You should know the Taylor series, at $a = 0$, of the following six functions. Ideally, you should remember how to derive them, as well:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots \\
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \\
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots
\end{aligned}$$