# MATH 8 CLASS 1 NOTES, SEPTEMBER 22 2010

## 1. Introductory Comments.

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Office phone: 646-2960, Office hours: Tuesday and Thursday, 12:30pm - 2:00pm, or by appointment (use email or talk to me in person.)
**Book**: *Calculus*, 6th edition, by James Stewart. Available at the bookstore.
**Webpage**: `www.math.dartmouth.edu/~m8f10`. This is very important as it will have a link to Webwork as well as a your weekly written homework assignments. It has comprehensive information about the class and will also be the place to go for updates as the class proceeds.
**Grading**: Based on homework and exams. By the end of the class, there will be four numerical scores, each out of 100: homework, midterm 1, midterm 2, and the final exam. If the final exam is your lowest score, your final raw score is obtained by summing your four scores together. If the final exam is not the lowest score, your lowest score is dropped and your final exam counts double (so out of 200). We then sort the class by score and fit the class to a curve to determine letter grades.

There are two types of homework: Webwork and written assignments. Webwork is an entirely computer-based homework system, where you login (go to the math 8 webpage for a link to the login page) using your ID and password, and get homework problems from the Webwork system. You input your answers back into the computer, and the system will tell you if you are right or wrong. If you are wrong, you have the chance (infinitely many chances, most of the time) to find the correct answer. Each set of problems in the Webwork system will have a 'closing time', after which you will not be able to work on the problems for credit anymore. You will get a new Webwork assignment each class, so be sure to check the system daily, or at least every other day! Usually, Webwork assignments will be due at 10am about half a week after they are originally assigned.

If you do not have a Webwork ID and password by Thursday, email me with your name and student ID number immediately!

Written homework assignments are more traditional and will be due once a week, on Monday at the beginning of class. You can turn them in at the beginning of class or at my office (slide under my door if I'm not in), before 12:30pm. Each week's homework will be listed on the course webpage. A grader will grade your assignments and they will usually be returned back in the homework boxes, no later than one week after they are due. On written assignments, we expect you to show all your work in a reasonably organized way. Correct answers without supporting justification will not be given full marks. Written homework and Webwork will each comprise half of your homework score.
**Tutorials**: On Sunday, Tuesday, and Thursday nights at 7:00pm - 9:00pm, three graduate students will run tutorial sessions for this class in Wilder 111. These sessions are informal and can be thought of as help sessions, and you are free to arrive or leave at any time.
**X-hour**: The X-hour for this section is on Tuesday, at 1:00pm - 1:50pm. Keep this slot of time available – although we will not use it regularly, there will be at least a few weeks where we will use the X-hour as a replacement class for another day where I will not be in.
**Late homework policy**: In general, unexcused late homework will not be accepted for a score. The only general reasons we will grant extensions on homework are for illness or family emergencies. In these cases, please notify me before the assignment is due with the

reason why you cannot turn in the assignment on time. If you have some other reason why you cannot finish an assignment on time, you can always email me and ask for an extension, although I cannot guarantee that you receive one. This late policy homework applies to both Webwork and written homework.

**Assistance**: In general, there is a lot of assistance available for this class. There are the previously mentioned tutorial sessions, office hours, either for me or for the other two instructors for math 8, and also the Tutor Clearinghouse. The last option might require you to pay a small fee, although some students have found it useful and worth the price in the past.

If you are having trouble in the class, do not hesitate to seek help.

## 2. General advice

- The most important piece of advice is to keep up with the progress of the class. Mathematics may very well be the subject where progress at any one point is most dependent on understanding everything that came before it, so once you fall behind you will have difficulty understanding the material in subsequent classes.
- The best way to test yourself for understanding of mathematical content is to solve problems by yourself. Of course, if you are having some difficulty and are not making progress on a problem, you should feel free to seek assistance from classmates, TAs, or the instructors, but it is worth trying each problem on your own for at least some amount of time. Even if you do not find a solution you may benefit from partial progress on the problem, or discover your mathematical weaknesses.
- If you are really serious about learning mathematics well, it might be a good idea to work problems from the textbook which are not assigned (and there will be many of these). In particular, you want to look at questions which are difficult, since those are the ones which will bring the greatest understanding.
- The above being said, for the best understanding of mathematical material, you should not only work homework assignments, but also discuss mathematics (with peers, TAs, or instructors) verbally. You will find that speaking, listening, reading, and writing mathematics are all a bit different from each other, and that maximal understanding only comes about when you engage in all four of these activities.
- Browse the section of a textbook that will be covered in class prior to actually attending the class. You shouldn't expect complete understanding, but some exposure to the terms and ideas before attending class will probably make class more enlightening and less confusing.
- If you start to have difficulty, do not hesitate to seek help. As mentioned earlier, falling behind is highly undesirable, and there is plenty of available assistance.
- Nevertheless, don't necessarily expect to understand everything at once. Repeated exposure to the same material will build understanding, and something which seemed incomprehensible at first glance can become understandable with repeated effort.
- Try to do a little bit of mathematics everyday. Of course, you really don't have a lot of choice since there are homework assignments every other day, but doing mathematics isn't very different from playing a sport or a musical instrument – unless you have a lot of experience, you need to practice daily to stay at your best. We don't expect you to work on math two hours a day, but something more like thirty minutes per day, on average (outside of lectures, of course) isn't unreasonable.

## 3. Chapter 12.1: Sequences

3.1. **Introduction.** Informally speaking, a sequence is a list of numbers, presented with an ordering. We will often designate the members of a sequence with the variables $a_n, b_n$, etc, where the $n$ (usually) range over positive or non-negative integers.

**Examples.**

- Let $a_1 = 1, a_2 = 2, a_3 = 3, \ldots$, and in general, $a_n = n$.
- Let $a_2 = 1, a_3 = 1, \ldots$, and in general, $a_n = 1$. Notice that a sequence does not have to start with an $a_1$ term; in this example we start with $a_2$, and we could also have started with $a_0$ or even $a_{-1}$.
- Let $a_0 = 1, a_1 = 1/2, a_2 = 1/4, \ldots$, and in general, $a_n = 1/2^n$.

Notice that in each of these examples, we were able to define the general term of a sequence using an explicit function of a real variable: $f(x) = x, 1$, and $1/2^x$, respectively. This does not always have to be the case, as the following two examples show:

**Examples.**

- Let $a_1 = 1, a_2 = 2 \cdot 1 = 2, a_3 = 3 \cdot 2 \cdot 1 = 6, a_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 24, \ldots$, and in general, $a_n = n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$. We call this the factorial function. Notice that this function has no obvious extension to non-integer values, or even negative integer values. We also define $0! = 1$. The factorial function will be very important in the subsequent parts of this chapter.
- Let $a_1 = 1, a_2 = 1$, and for $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$. Therefore, the first few terms of this sequence are $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5, a_6 = 8, a_7 = 13, a_8 = 21, \ldots$. This is what is known as a *recursive definition*, where a general term of the sequence is defined using the preceding terms. This particular sequence is known as the Fibonacci sequence. Notice that there is no obvious explicit formula for the $n$th term of the sequence.

From a standpoint of notation, we may write $\{a_n\}$ for the sequence composed of all the elements $a_n$. If we want to know when this sequence starts, we may write something like $\{a_n\}_{n=1}^{\infty}$, say, to indicate that the sequence starts with $a_1$.

One way to visualize a sequence is to plot its elements by pretending that the sequence is a function. Unlike a function of a real variable, the plot of a sequence is a series of dots. For example, we can plot each of the above sequences by drawing dots at $(n, a_n)$ for each $n$.

3.2. **Limits.** One of the most important properties of a sequence is whether or not it has a limit. This term is familiar from Calculus I, and it turns out that the notion of the limit of a sequence is very similar to the notion of the limit of a function.

Intuitively speaking, we say the sequence $\{a_n\}$ converges to a limit $L$ ($L$ is a real number) if the terms $a_n$ of the sequence approach $L$ as $n$ gets large. If a sequence does not converge to some limit, then we say the sequence diverges. If a sequence $\{a_n\}$ converges to $L$, we will frequently write this as

$$\lim_{n \to \infty} a_n = L$$

Notice that, written this way, there is almost no difference between this definition and the definition of the limit of a function $f(x)$ as $x \to \infty$. As a matter of fact, if we have a sequence $a_n$ which arises from a function $f(x)$, in the sense that $a_n = f(n)$ for all (positive) integers $n$, then $\lim_{n \to \infty} a_n = L$ if $\lim_{x \to \infty} f(x) = L$. (This is Theorem 12.1.3 of the text.) A more precise definition of the limit of a sequence, using epsilons, is

**Definition.** A sequence $a_n$ converges to the limit $L$ if, for all $\varepsilon > 0$, there exists some $N$ such that for all $n > N$, $|a_n - L| < \varepsilon$.

The analogy between limits of sequences and limits of functions does not end here. Recall that the limit of a sum of functions is the sum of the limits, etc. The same holds true for sequences: namely, we have identities like

$$\begin{aligned} \lim_{n\to\infty}(a_n + b_n) &= \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \\ \lim_{n\to\infty}(a_n b_n) &= \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n \end{aligned}$$

whenever $\{a_n\}$, $\{b_n\}$ are both convergent sequences. There are other similar properties; see page 714 of the text for more details.

If a sequence is described by an explicit formula, we can often use techniques from Calculus I to determine whether the sequence converges or diverges, and find the limit.

**Examples.**

- Suppose $a_n = (\sin n)/n$. Does this sequence converge or diverge, and if it converges, to what limit? If we consider the function $f(x) = \sin x/x$, then we quickly see that $\lim_{x\to\infty} f(x) = 0$, since the numerator is bounded (fluctuates between $-1$ and 1), while the denominator goes to infinity. Therefore, the sequence $\{a_n\}$ converges to the limit 0.
- Suppose $a_n = \sqrt{n}/(2\sqrt{n}+1)$. The function $f(x) = \sqrt{x}/(2\sqrt{x}+1)$ has limit 1/2 as $x \to \infty$, as one can see by dividing both numerator and denominator by $\sqrt{x}$ and then taking the limit.
- Suppose $a_n = (\ln n)/n$. The function $f(x) = \ln x/x$ has limit 0 as $x \to \infty$, as an application of L'Hopital's rule will show. Therefore, the sequence $\{a_n\}$ converges to the limit 0.
- Suppose $a_n = \cos(1/n)$. The function $f(x) = \cos(1/x)$ has limit 1 as $x \to \infty$, since $1/x \to 0$, and $\cos 0 = 1$. Therefore, the sequence $\{a_n\}$ converges to the limit 1.

Another way to evaluate the last limit is to use the property that if $f(x)$ is a function continuous at $L$, and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(L)$. So in the above example, we are using the fact that $\lim_{n\to\infty} 1/n = 0$, and that $\cos x$ is continuous at $x = 0$. Can you think of a counterexample to this property if $f(x)$ is not continuous at $L$?

Another useful property to know (and one which probably seems self-evident) is the following: if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$. For instance, this shows that $a_n = (-1)^n/n$ converges to 0. It's important to remember that this can definitely be false if the limit of $|a_n|$ is not equal to 0. Again, can you think of a counterexample?

In all of the previous examples, the sequences we have examined have converged. Of course, sequences can also diverge:

**Examples.**

- Let $a_n = n$. Then this sequence evidently diverges. Notice that as $n \to \infty$, $a_n \to \infty$ as well. Whenever this is the case, we sometimes write $\lim_{n\to\infty} a_n = \infty$; however, we still call $\{a_n\}$ a divergent sequence and sometimes say that the sequence diverges to infinity. When we say that a sequence diverges to infinity, we really mean that all, and not some of, the terms get very large as $n$ gets large.
- Let $a_n = 1$ if $n$ is odd, and $a_n = n$ if $n$ is even. So the sequence starts out $1, 2, 1, 4, 1, 6, \ldots$. Does this sequence diverge? If so, does it diverge to infinity? (Talk about this with your neighbor.) This sequence does diverge, but not to infinity. It

evidently does not have a limit, but it does not diverge to infinity because no matter how far out in the sequence we get, there are still some elements which are small (namely, all the odd-indexed terms).

- Let $a_n = n!$. This sequence also diverges to infinity, since $n!$ grows very rapidly (certainly faster than $a_n = n$).
- Let $a_n = (-1)^n$. This sequence is given by $\{-1, 1, -1, 1, \ldots\}$. Notice that although this sequence does not get arbitrarily large, it still diverges since its terms do not approach any fixed value.

Another convenient (and probably self-evident) property is the following. If $\lim_{n\to\infty} a_n = \infty$, then $\lim_{n\to\infty} 1/a_n = 0$. Intuitively, if $a_n$ is getting arbitrarily large as $n \to \infty$, we expect $1/a_n$ to be arbitrarily small. So this shows, for instance, that $\lim_{n\to\infty} 1/n! = 0$. We will see another way of calculating this limit soon.

It is useful to note that the convergence or divergence of a sequence is not impacted by the behavior of a finite number of terms of the sequence. For instance, maybe we have a sequence which starts with $1, 2^2, 3^{3^3}, 4^{4^{4^4}}$. Even though the fourth term is already ridiculously large (it has more digits than what is believed to be the number of atoms in the universe), if the remaining terms are given by $1/n, n \geq 5$, then the sequence is still going to converge to 0.

A special class of sequences which we sometimes consider are monotonic sequences. We say that a sequence $\{a_n\}$ is increasing if $a_1 < a_2 < a_3 < \ldots$, while a sequence is decreasing if $a_1 > a_2 > a_3 > \ldots$. A sequence is monotonic if it is either decreasing or increasing.
The fact that a sequence is decreasing or increasing does not tell us, on its own, whether the sequence converges or diverges:

**Examples.**

- In the above example $a_n = n$, the sequence $\{a_n\}$ is increasing and diverges to infinity.
- Let $a_1 = 1/2, a_2 = 3/4, a_3 = 7/8, \ldots$, and in general, $a_n = 1 - 1/2^n$. This is an increasing sequence, but converges to the limit 1.

However, there is an additional property which, in conjunction with monotonicity, can tell tell us that a sequence converges. We say that a sequence $\{a_n\}$ is bounded above if all of its terms are bounded by some number; that is, we can find a number $M$ such that $a_n < M$ for all $n$. Similarly, we say a sequence is bounded below if we can find a number $m$ such that $a_n > m$ for all $n$, and we say a sequence is bounded if it is both bounded above and below.

An obvious difference between the two examples above is that $a_n = n$ is not bounded (it gets arbitrarily large), while $a_n = 1 - 1/2^n$ is bounded (below by 0, above by 1). We might conjecture that bounded, monotonic sequences are convergent; as a matter of fact, this is true:

**Theorem.** (Monotonic Sequence Theorem) Any bounded, monotonic sequence is convergent.

Of course, not every convergent sequence is monotonic (although they are all bounded). Can you think of an example of a convergent sequence which is not monotonic?

Another tool which is sometimes useful (and often used implicitly) is the squeeze theorem:

**Theorem.** (Squeeze Theorem). Suppose $a_n, b_n, c_n$ satisfy $a_n \leq b_n \leq c_n$ for all large $n$ (for all $n \geq N$, for some fixed $N$), and $\{a_n\}, \{c_n\}$ both converge to the same limit $L$. Then $\{b_n\}$ converges to $L$ as well.

Let's conclude with some additional examples, all involving factorials:

**Examples.**

- Let $a_n = 1/n!$. We already saw that the limit is equal to 0. Another way of showing this is to apply the squeeze theorem. For example, we always have $0 \leq 1/n! \leq 1/n$, and the sequences $\{0\}$, $\{1/n\}$ both converge to 0, so the squeeze theorem tells us $\lim_{n\to\infty} 1/n! = 0$ as well.
- Let $a_n = n^2/n!$. We can either apply the squeeze theorem or rewrite this sequence and use limit laws. For the first approach, notice that $0 \leq a_n \leq 2/(n-2)!$ for all $n \geq 3$, since
$$\frac{n^2}{n!} = \frac{n \cdot n}{n \cdot n - 1 \cdots 1} = \frac{n}{n} \cdot \frac{n}{n-1} \cdot \frac{1}{(n-2)!}$$
and $n/n - 1 \leq 2$. Using the fact that $2/(n-2)!$ converges to 0 as $n \to \infty$, the squeeze theorem tells us that $\lim_{n\to\infty} n^2/n! = 0$. For the other approach, we have
$$\lim_{n\to\infty} \frac{n^2}{n!} = \lim_{n\to\infty} \left(\frac{n^2}{n(n-1)}\right)\left(\frac{1}{(n-2)!}\right) = 1 \cdot 0$$
A bit of thought shows that there was nothing special about the exponent '2' in the numerator. We could have used any arbitrarily large exponent like $n^{1000000}$ and still have gotten the same result; namely, convergence to the limit 0. This shows that the factorial function grows faster than any power of $n$.