Dartmouth College
Mathematics 81

1. p 567: #6, 7

2. pp 581-2: #3, 4, 9

3. Let K/F be a finite separable extension.

   (a) Show that there is a "smallest" finite extension L of K with L/F Galois. L is called the Galois closure of K/F.

   (b) Determine the Galois closure L of $\mathbb{Q}(\sqrt[3]{2}, \sqrt[5]{2})/\mathbb{Q}$, and compute its degree over $\mathbb{Q}$.

   (c) For L as in the previous part, determine whether or not $Gal(L/\mathbb{Q})$ is abelian. Hint: You certainly can do this without computing the group explicitly, i.e. "brain" versus "brawn".

4. Suppose that K/F is a finite Galois extension of degree n with Galois group $G = \{\sigma_1, \ldots, \sigma_n\}$. For an element $\alpha \in K$, define its norm and trace as follows:

$$Tr_{K/F}(\alpha) = \sigma_1(\alpha) + \cdots + \sigma_n(\alpha)$$
$$N_{K/F}(\alpha) = \sigma_1(\alpha)\sigma_2(\alpha)\cdots\sigma_n(\alpha)$$

   (a) Show that $Tr_{K/F}$ and $N_{K/F}$ map K to F, and satisfy $Tr_{K/F}(\alpha+\beta) = Tr_{K/F}(\alpha) + Tr_{K/F}(\beta)$ and $N_{K/F}(\alpha\beta) = N_{K/F}(\alpha)N_{K/F}(\beta)$ for all $\alpha, \beta \in K$.

   (b) Show that $Tr_{K/F}$ is a surjective. Hint: first show that there is an element $\alpha \in K$ for which $Tr_{K/F}(\alpha)$ is not zero. Note that in characteristic 0 or characteristic p with p not dividing n, this is very easy, but there is a general way to do this in all cases.