Dartmouth College
Mathematics 81

Homework 3: Due on Wednesday, January 30.

1. Let F be a field of characteristic 0, and let m and n be distinct integers with $\sqrt{m} \notin F$, $\sqrt{n} \notin F$, and $\sqrt{mn} \notin F$.

   (a) Show that $[F(\sqrt{m}, \sqrt{n}) : F] = 4$.

   (b) Show by example that the above proposition is false if we only assume that $\sqrt{m} \notin F$ and $\sqrt{n} \notin F$.

2. Let $m_1, m_2, \ldots, m_t$ be distinct integers none of which are squares in $\mathbb{Z}$.

   (a) Show that $[\mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_t}) : \mathbb{Q}] \le 2^t$, and give an example to show that the inequality can be strict.

   (b) Now assume that the integers are square-free and relatively prime in pairs. Show that $[\mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_t}) : \mathbb{Q}] = 2^t$. Hint: Induction on t and problem 1 may be of use.

3. Consider the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q}$. Determine a basis for this extension. Hint: Rather than trying to prove directly that the set you write down is linearly independent, give an argument, based on what we have done in class, which proves your set is a basis. Your argument should extend easily to $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7})/\mathbb{Q}$ for which you should simply state (with confidence) a basis.

4. Determine the degree of the extension $\mathbb{Q}(i, \sqrt{3}, e^{2\pi i/3})/\mathbb{Q}$, and write down three intermediate fields K, (i.e, with $\mathbb{Q} \subsetneq K \subsetneq \mathbb{Q}(i, \sqrt{3}, e^{2\pi i/3})$).

5. (19, p531) Let K/F be a field extension of degree n.

   (a) For any $\alpha \in K$ show that left multiplication by α is an F-linear transformation on K.

   (b) Prove that K is isomorphic to a subfield of $M_n(F)$. In particular this shows that $M_n(F)$ contains an isomorphic copy of all field extensions of F having degree $\le n$.

6. (20, p531) For T_α as defined above (and $F = \mathbb{Q}$), show that α is a root of the characteristic polynomial of T_α, and use this to find (different) monic polynomials of degree 3 of which $\sqrt[3]{2}$ and $1 + \sqrt[3]{2} + \sqrt[3]{4}$ are roots.

7. (21, p531) Let $K = \mathbb{Q}(\sqrt{D})$ for some squarefree integer D, and let $\alpha = a + b\sqrt{D}$ be an element of K. Let $\mathcal{B} = \{1, \sqrt{D}\}$ be a basis for $K/\mathbb{Q}$ and show that the matrix of T_α with respect to $\mathcal{B}$ is $\begin{pmatrix} a & bD \\ b & a \end{pmatrix}$. Prove directly that the map $a + b\sqrt{D} \mapsto \begin{pmatrix} a & bD \\ b & a \end{pmatrix}$ is an isomorphism of K onto a subfield of $M_2(\mathbb{Q})$.