Dartmouth College

Mathematics 81

The following exercises are intended to remind you of (or sharpen your skills regarding) material from Math 71. When convenient we shall denote the quotient ring $\mathbb{Z}/m\mathbb{Z}$ by $\mathbb{Z}_m$. Also, recall that all our ring homomorphisms take the multiplicative identity of one ring to the multiplicative identity of the other. Below are a few handy results which you may use without proof, but if they are unfamiliar, you should read the relevant material in your text.

Let $f \in \mathbb{Z}[x]$. f is called *primitive* if and only if the gcd of its coefficients is 1. For example $2x^2 - 6x + 3$ is primitive in $\mathbb{Z}[x]$. The following two theorems are essentially (if not in fact) equivalent to Gauss's lemma over $\mathbb{Q}$:

Theorem: Let $f \in \mathbb{Z}[x]$. Then f is irreducible in $\mathbb{Z}[x]$ if and only if f is primitive in $\mathbb{Z}[x]$ and irreducible in $\mathbb{Q}[x]$.

Theorem: Let $f \in \mathbb{Z}[x]$, and suppose that $f = gh$ for two polynomials $g, h \in \mathbb{Q}[x]$. Then $f = g_0h_0$ for polynomials $g_0, h_0 \in \mathbb{Z}[x]$ with $\deg(g) = \deg(g_0)$ and $\deg(h) = \deg(h_0)$. In particular g_0 and h_0 are rational scalar multiples of g and h respectively.

1. Show that there exist ring homomorphisms $\mathbb{Z}_m \to \mathbb{Z}_n$ if and only if $n \mid m$. **Hint:** The "zeroth homomorphism" theorem makes one implication almost effortless. Show that all such homomorphisms must be surjective.

2. For each of the ideals I listed below, determined whether the ideal I is prime, maximal, or neither in $\mathbb{Z}[x]$ and then in $\mathbb{Q}[x]$ by examining the appropriate quotient ring. If the quotient is not an integral domain, find zero divisors. If the quotient is not a field, then I is not maximal, so find a maximal ideal M with $I \subsetneq M$, and justify that M is maximal.

   (a) $I = (x^3 + 2)$
   (b) $I = (5, x^3 + 2)$
   (c) $I = (7, x^3 + 2)$

3. More examples in $\mathbb{Z}[x]$.

   (a) Which (if any) of $(x^3 + 2, x^3 + 9)$ or $(x^3 + 2, x^3 + 7)$ are maximal ideals in $\mathbb{Z}[x]$?
   (b) Find infinitely many maximal ideals containing $(x^3 + x^2)$.

4. Let f be a nonconstant polynomial in $\mathbb{Q}[x]$. Show that there are only finitely many maximal ideals in $\mathbb{Q}[x]$ containing (f).

5. It follows from the proof of Theorem 14 in section 8.3 that every proper ideal in $\mathbb{Q}[x]$ is contained in a maximal ideal (see p 288). That fact remains true for $\mathbb{Z}[x]$ as well, but for reasons not typically covered in math 71.

   Proof or counterexample: Let P be a nonzero prime ideal in $\mathbb{Z}[x]$, and I an ideal with $P \subseteq I \subsetneq \mathbb{Z}[x]$. Then I is a prime ideal.