Dartmouth College
Mathematics 81

This problem is part of the assignment due on Wednesday, 14 January.

The following exercises are to remind you of (or sharpen your skills regarding) material from Math 71. First, I give a few results which you may use without proof, but if they are unfamiliar, you should read the relevant material in your text.

Let $f \in \mathbb{Z}[x]$. f is called *primitive* if and only if the gcd of its coefficients is 1. For example $2x^2 - 6x + 3$ is primitive in $\mathbb{Z}[x]$. The following two theorems are essentially (if not in fact) equivalent to Gauss's lemma over $\mathbb{Q}$:

Theorem: Let $f \in \mathbb{Z}[x]$. Then f is irreducible in $\mathbb{Z}[x]$ if and only if f is primitive in $\mathbb{Z}[x]$ and irreducible in $\mathbb{Q}[x]$.

Theorem: Let $f \in \mathbb{Z}[x]$, and suppose that $f = gh$ for two polynomials $g, h \in \mathbb{Q}[x]$. Then $f = g_0 h_0$ for polynomials $g_0, h_0 \in \mathbb{Z}[x]$ with $\deg(g) = \deg(g_0)$ and $\deg(h) = \deg(h_0)$. In particular g_0 and h_0 are integer scalar multiples of g and h respectively.

Here are the problems to work on:

1. Consider the following **false** statement. Let $f \in \mathbb{Z}[x]$ with $\deg(f) = 2$ or 3. Then f is irreducible in $\mathbb{Z}[x]$ if and only if f has no roots in $\mathbb{Q}$.

   One direction is true; prove that direction quoting nontrivial results from your text. Find a counterexample for the converse. Then fix the statement (in some nontrivial way) so that the new statement is true in both directions. You should justify why your new converse is true.

2. The following exercise is meant to refresh your memory about ideals and quotient rings. For each of the ideals I listed below, determined whether the ring $\mathbb{Z}[x]/I$ has zero divisors, is an integral domain, or is a field (and hence whether the ideal I is not prime, prime, or maximal). If the quotient is not an integral domain, find zero divisors. If the quotient is not a field, then I is not maximal, so find a maximal ideal M with $I \subsetneq M$, and justify that M is maximal.

   Hint: You may use without proof the very handy fact that if $f \in \mathbb{Z}[x]$ and $m \in \mathbb{Z}$, then $\mathbb{Z}[x]/(m, f(x)) \cong (\mathbb{Z}/m\mathbb{Z})[x]/(\bar{f}(x))$, where $\bar{f}(x)$ is the polynomial in $(\mathbb{Z}/m\mathbb{Z})[x]$ obtained from f by reducing the coefficients modulo m.

   (a) $I = (x^3 + 2)$
   (b) $I = (5, x^3 + 2)$
   (c) $I = (7, x^3 + 2)$