Edge-Augmented Fourier Partial Sums with Applications to Magnetic Resonance Imaging (MRI)

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ABSTRACT

Certain applications such as Magnetic Resonance Imaging (MRI) require the reconstruction of functions from Fourier spectral data. When the underlying functions are piecewise-smooth, standard Fourier approximation methods suffer from the *Gibbs phenomenon* – with associated oscillatory artifacts in the vicinity of edges and an overall reduced order of convergence in the approximation. This paper proposes an edge-augmented Fourier reconstruction procedure which uses only the first few Fourier coefficients of an underlying piecewise-smooth function to accurately estimate jump information and then incorporate it into a Fourier partial sum approximation. We provide both theoretical and empirical results showing the improved accuracy of the proposed method, as well as comparisons demonstrating superior performance over existing state-of-the-art sparse optimization-based methods.

Keywords: Fourier Reconstruction, Gibbs Phenomenon, Edge Detection, MR Imaging

1. INTRODUCTION

This paper addresses the problem of reconstructing a 2π-periodic piecewise-smooth function f given its $2N+1$ lowest frequency Fourier series coefficients,

$$\hat{f}_k := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx, \qquad k \in [-N, N] \cap \mathbb{Z}. \tag{1}$$

The Fourier partial sum reconstruction of such piecewise-smooth f,

$$S_N f(x) := \sum_{|k|\leq N} \hat{f}_k e^{ikx}, \qquad x \in [-\pi, \pi), \tag{2}$$

suffers from the *Gibbs phenomenon*[1] – with its associated non-physical oscillations in the vicinity of jump discontinuities, and an overall reduced order of accuracy in the reconstruction. In applications such as MR imaging – where the scanning apparatus collects Fourier coefficients[2] of the specimen being imaged – these oscillatory artifacts and the reduced order of accuracy are significant impediments to the rapid generation of accurate images. Hence, there exists significant ongoing and inter-disciplinary interest in novel methods of reconstructing such functions from Fourier spectral data.

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1.1 Related Work

The traditional approach to mitigating Gibbs artifacts is using low-pass filtering.[1] However, this does not completely eliminate all artifacts; filtered reconstructions still suffer from smearing in the vicinity of edges and improved convergence rates are restricted to regions away from edges. Spectral reprojection methods such as[3] work by reconstructing the function in each smooth interval using an alternate (non-periodic) basis such as those consisting of Gegenbauer polynomials. While these methods have been shown to be highly accurate, they are sensitive to parameter choice; indeed, small errors in parameter selection or estimated edge location (which are used to determine the intervals of smoothness) can lead to large reconstruction errors. More recently, there has been significant interest in compressed sensing[4,5] based approaches to this problem.[6,7] While these approaches are indeed extremely powerful and versatile, they are implicitly discrete methods and do not perform well when provided with continuous measurements as is the case here. Indeed, we show in the empirical results below that they exhibit poor (first-order) numerical convergence when recovering a piecewise-smooth function f from its *continuous* Fourier measurements (1). The method proposed in this paper is perhaps most closely related to those in[8–14] all of which use prony-like methods to estimate the jumps in f (after which the jumps' deleterious spectral effects can be mitigated). In contrast to these previous approaches, however, herein we (i) propose the use of a simple spectral extrapolation scheme together with an alternate (and highly noise robust) non-prony-based jump estimation procedure, (ii) provide (ℓ_2-norm) error analysis of the proposed method for general piecewise-smooth functions, (iii) present comparisons with state-of-the-art sparse optimization based reconstruction methods, *and* (iv) present preliminary 2D reconstruction results.

The rest of this paper is organized as follows: In §2, we relate jump information to the Fourier coefficients of a piecewise-smooth function. We then show that this jump information can be incorporated into a modified Fourier partial sum approximation. We also briefly summarize two methods of accurately estimating jump information when given only Fourier spectral data. §3 provides theoretical error bounds for the proposed reconstruction method while §4 presents empirical results (and comparisons to existing methods) demonstrating the improved accuracy.

2. INCORPORATING JUMP INFORMATION

We begin by first relating jump information to the Fourier coefficients of a piecewise-smooth*function f. Let f have a finite number of jump discontinuities at $-\pi < x_1 < \cdots < x_J \leq \pi$. Furthermore, define the *jump function*, $[f]$, associated with f as follows:

Definition 2.1. *The* jump function *of a piecewise-continuous function f is defined by*

$$[f](x) := f(x^+) - f(x^-)$$

for all $x \in \mathbb{R}$, where $f(x^-)$ and $f(x^+)$ are the left- and right-hand limits of f at x respectively. The value $[f](x)$ will sometimes be referred to as the jump height *at x.* By using integration by parts on the Fourier integral (1) and the above definition, one can show the following result:

Theorem 2.1. *If $f : \mathbb{R} \to \mathbb{R}$ is piecewise smooth then*

$$\left| \hat{f}_k - \sum_{j=1}^{J} \frac{[f](x_j)}{2\pi i k} e^{-ikx_j} \right| \leq \frac{C}{k^2}$$

holds for all $k \in \mathbb{Z} \setminus \{0\}$, where $C \in \mathbb{R}^+$ is an absolute constant that only depends on f'' and the jump function of f'. As a result, both $\hat{f}_k = \mathcal{O}(1/|k|)$ and $\hat{f}_k \sim \sum_{j=1}^{J} \frac{[f](x_j)}{2\pi i k} e^{-ikx_j}$ are true for all $k \in \mathbb{Z} \setminus \{0\}$. For brevity, we omit proofs of the analytical results presented here. The interested reader is referred to[15] for all proofs and additional discussion.

*For specifics of the definition of piecewise-continuity and piecewise-smoothness used in this discussion, see.[15]

Assuming that the jump locations $x_1, \dots, x_J$ and jump heights $[f](x_1), \dots, [f](x_J)$ of f are known, one can use Theorem 2.1 to estimate the Fourier coefficients of f for all $k \in \mathbb{Z} \setminus \{0\}$ by

$$\hat{f}_k^{est} := \sum_{j=1}^{J} \frac{[f](x_j)}{2\pi i k} e^{-ikx_j}. \tag{3}$$

We can now augment (2) by incorporating these jump-based Fourier coefficient estimates. The resulting augmented partial sum approximation, $S_N^{edge} f$, is defined by

$$S_M^{edge} f(x) := \sum_{|k| \leq N} \hat{f}_k e^{ikx} + \sum_{|k|>N} \hat{f}_k^{est} e^{ikx} \tag{4}$$

for all $x \in \mathbb{R}$. Note that $S_N^{edge} f$ still only utilizes $2N+1$ true Fourier coefficients of f.

We would like to use as many terms from the last sum in (4) as we can. Toward this end, let's consider the form of the complete last sum with $\hat{f}_0^{est} := 0$. It is easy to show that

$$\sum_{k=-\infty}^{\infty} \hat{f}_k^{est} e^{ikx} = \sum_{j=1}^{J} [f](x_j) \left(\sum_{0<|k|<\infty} \frac{e^{-ikx_j}}{2\pi i k} e^{ikx} \right).$$

Note that the k^{th} Fourier coefficient of the 2π-periodic *ramp function* $r_j(x)$, defined by

$$r_j(x) := \begin{cases} \frac{-\pi - x}{2\pi}, & x < x_j \\ \frac{\pi - x}{2\pi}, & x > x_j \end{cases} \tag{5}$$

for all $x \in [-\pi, \pi)$, is given by $(\widehat{r_j})_k = \frac{e^{-ikx_j}}{2\pi i k}$ for all $k \in \mathbb{Z} \setminus \{0\}$. Also, $(\widehat{r_j})_0 = 0$. Thus, we have that

$$\sum_{k=-\infty}^{\infty} \hat{f}_k^{est} e^{ikx} = \sum_{j=1}^{J} [f](x_j) r_j(x).$$

We are now able to give a more easily computable closed form expression for $S_N^{edge} f$ by noting that

$$S_N^{edge} f(x) = \sum_{k=-N}^{N} (\hat{f}_k - \hat{f}_k^{est}) e^{ikx} + \sum_{j=1}^{J} [f](x_j) r_j(x). \tag{6}$$

2.1 Estimating Edge Information from Fourier Data

The above reconstruction for $S_N^{edge} f$ assumes we have exact or accurate estimates of jump locations and jump values. While this may be available in certain applications, it is more likely that we have to estimate edge information given Fourier data. One simple approach (a variant of which is utilized, e.g., in[8]) is to use Prony's method.[16,17] Multiplying (3) by $2\pi i k$, we obtain a weighted complex exponential sum, from which the jump locations and values can be estimated using Prony's method.

This Prony-based approach suffers from certain drawbacks; for example, it requires apriori estimates for the number of jumps, and it is not very tolerant of measurement errors. Alternatively, we propose use of the *concentration kernel*[18–20] method of edge detection which uses specially chosen "filter" factors known as *concentration factors* and Fourier partial sums to approximate the jump function $[f]$. The interested reader is referred to[18–20] for details, as well as[21] for a statistical detection-theoretic analysis of the method.

3. ERROR ANALYSIS

If f is piecewise smooth then $\|f - S_N f\|_2 = \mathcal{O}(\sqrt{1/N})$.[1] In this section we show that incorporating edge information allows us to do better. In particular, if we have access to true jump location and height information we can form $S_N^{edge} f$ from (6) and achieve reduced errors.

THEOREM 3.1. *If f is piecewise smooth then*

$$||f - S_N^{edge} f||_2 \leq \sqrt{\frac{2c^2}{3N^3}}$$

for some constant $c \in \mathbb{R}^+$ which is independent of N.

Of course, in practice one does not have access to true jump location and height information. In such settings r_j of (5) can not be computed exactly. Let $\widetilde{r_j}$ denote another ramp function with a jump at $\widetilde{x_j}$ instead of x_j, and with an associated magnitude of $a_j \approx [f](x_j)$. We will assume in this section that $\widetilde{r_j}(x)$ is an approximation of $r_j(x)$ produced using only the first $2N+1$ Fourier coefficients of f by one of the methods in §2.1. These approximate ramp functions can then be used to build an estimated jump-information based approximation of f given by

$$\widetilde{f}(x) = \sum_{k=-N}^{N} (\hat{f}_k - \widetilde{\hat{f}_k^{est}}) e^{ikx} + \sum_{j=1}^{J} a_j \widetilde{r_j}(x), \tag{7}$$

where $\widetilde{\hat{f}_k^{est}}$ denotes the k^{th} Fourier coefficient of $\sum_{j=1}^{J} a_j \widetilde{r_j}$. Note that $\widetilde{f}$ is an approximation to $S_N^{edge} f$ which is still formed using only $2N+1$ true Fourier coefficients of f. The next theorem demonstrates that it will closely approximate f as long as the jump locations and heights are estimated accurately enough.

THEOREM 3.2. *Let f be a piecewise smooth function, $S_N^{edge} f$ be the edge-augmented Fourier sum approximation of f with true jump information (6), and $\widetilde{f}$ be the edge-augmented Fourier sum approximation of f with estimated jump information (7). Then, if $|\widetilde{x_j} - x_j| < \epsilon$ and $|a_j - [f](x_j)| < \delta$ both hold for all $j \in \{1, \ldots, J\}$, we have that*

$$||f - \widetilde{f}||_2 \leq \sqrt{\frac{2c^2}{3N^3}} + J\delta + \sqrt{\frac{\epsilon}{2\pi}} \left(J\delta + \sum_{j=1}^{J} |[f](x_j)| \right)$$

where $c \in \mathbb{R}^+$ is a constant independent of N, ϵ, and δ.

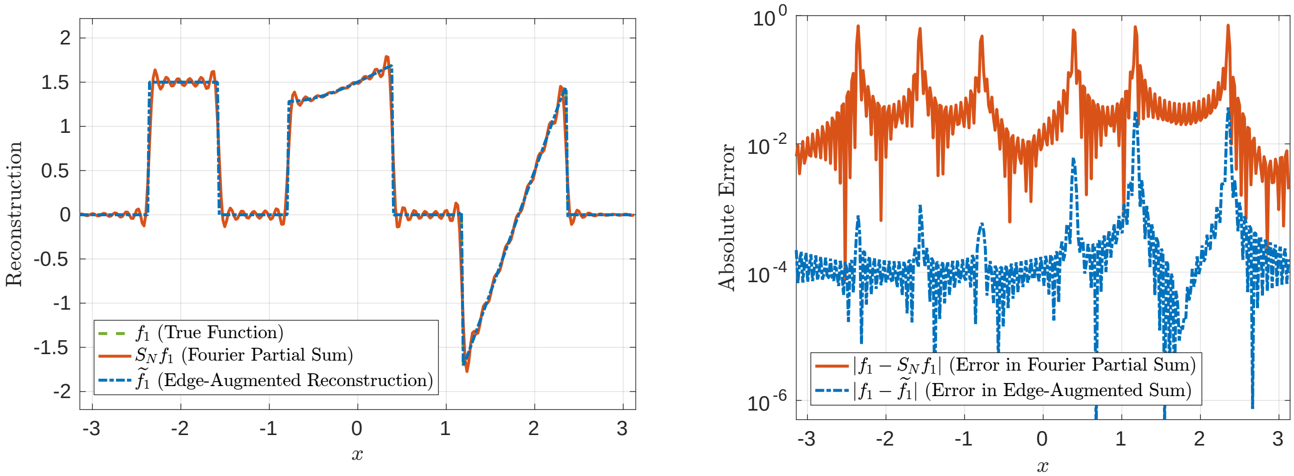
Comparing Theorem 3.2 to $\|f - S_N f\|_2 = \mathcal{O}(\sqrt{1/N})$ we can see that $\widetilde{f}$ from (7) will approximate piecewise smooth functions with jumps better than $S_N f$ will as long as δ is $o(1/\sqrt{N})$ and ϵ is $o(1/N)$.

4. EMPIRICAL RESULTS

Consider reconstruction of the piecewise-smooth function

$$f_1(x) = \begin{cases} \frac{3}{2} & \frac{-3\pi}{4} \leq x < -\frac{\pi}{2} \\ \frac{7}{4} - \frac{x}{2} + \sin(x - \frac{1}{4}) & \frac{-\pi}{4} \leq x < \frac{\pi}{8} \\ \frac{11}{4}x - 5 & \frac{3\pi}{8} \leq x < \frac{3\pi}{4} \\ 0 & \text{else} \end{cases} \tag{8}$$

given its first $|N| \leq 50$ Fourier series coefficients. Fig. 1(a) shows the reconstruction of f_1 using a standard Fourier partial sum $S_N f_1$ (solid line) as well as using the edge-augmented approximation $\widetilde{f_1}$ (dash-dot line) proposed in this paper. The concentration edge detection method was used to detect edge information for generating $\widetilde{f_1}$. All code used to generate the results in this section can be found at.[22] Fig. 1(b) then shows the associated absolute error in the respective reconstructions. We see that the proposed edge-augmented method is significantly more accurate, with pointwise errors often about 100 times smaller than a standard Fourier partial sum.



(a) Function Reconstruction

(b) Absolute Error in Reconstruction

Figure 1: Edge-Augmented Fourier Reconstruction of the Piecewise-Smooth Function f_1 using the first $|N| \leq 50$ Fourier Coefficients

Next, we confirm the analytical results of §3 by plotting the convergence rate of the proposed approximation in Fig. 2. Here, we plot the absolute reconstruction error as a function of the number of Fourier coefficients N for a standard Fourier partial sum $(+)$, the edge-augmented reconstruction with true edge information $(\circ)$ and the edge-augmented reconstruction with estimated edge information $(\times)$. Once again, the concentration edge detection procedure was used to estimate edge information. The dashed lines indicate theoretical convergence rates for reference. Not only does this plot confirm the improved convergence of the approximation, it also shows that using estimated edge information does not incur significant error.

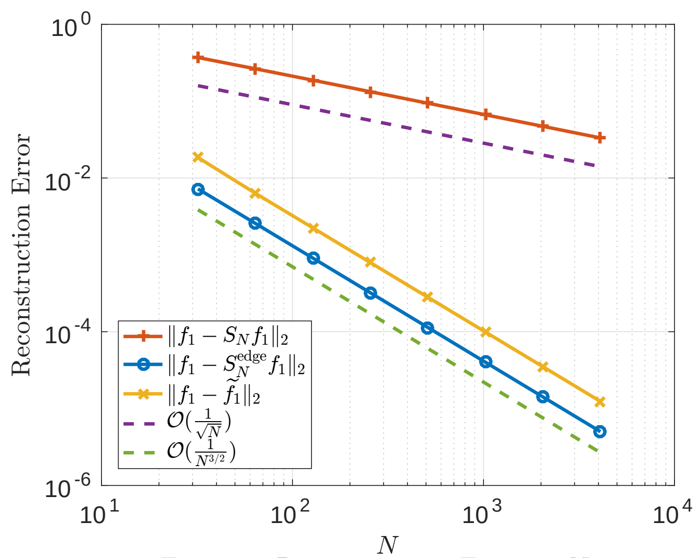
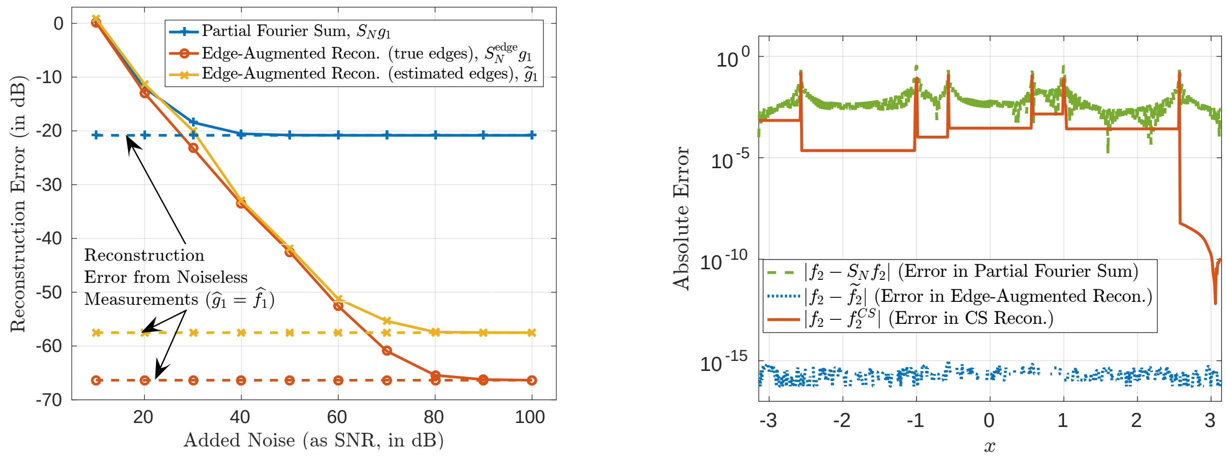


Figure 2: Reconstruction Error vs N

Fig. 3a investigates the robustness of the proposed method to measurement noise. The figure plots error (averaged over 50 trials, measured in dB) when reconstructing f_1 from noise corrupted Fourier coefficients,

$$(\widehat{g}_1)_k = (\widehat{f}_1)_k + n_k, \ k \in [-100, 100] \cap \mathbb{Z}, \ n_k \sim \mathcal{CN}(0, \sigma^2),$$

where σ^2 is chosen as per the desired SNR. For reference, the reconstruction error from noiseless measurements is indicated using dashed lines (note: this is non-negligible; for example, the partial Fourier sum suffers from Gibbs artifacts). We see that the proposed method (and the associated concentration edge detection procedure) is robust and significantly improves on the accuracy of standard Fourier reconstruction methods.



(a) Robustness to Measurement Noise – Error in reconstructing f_1 using noisy Fourier coefficients $\{(\widehat{g}_1)_k = (\widehat{f}_1)_k + n_k\}_{k=-100}^{100}$

(b) Comparison of Proposed Method with Compressed Sensing Based Reconstruction Procedures

Figure 3: Empirical Evaluation of the Proposed Method with Noisy Measurements and against other Reconstruction Methods

Finally, Fig. 3b compares the reconstruction error of the proposed method with a compressed sensing based procedure when reconstructing the (2π-periodic) piecewise-constant function

$$f_2(x) = \Pi(x/2) + 0.5\Pi(x/2 - \pi/4) - 0.5\Pi(x/2 + \pi/4), \tag{9}$$

where Π denotes the standard rectangular function by solving the following TV-minimization problem

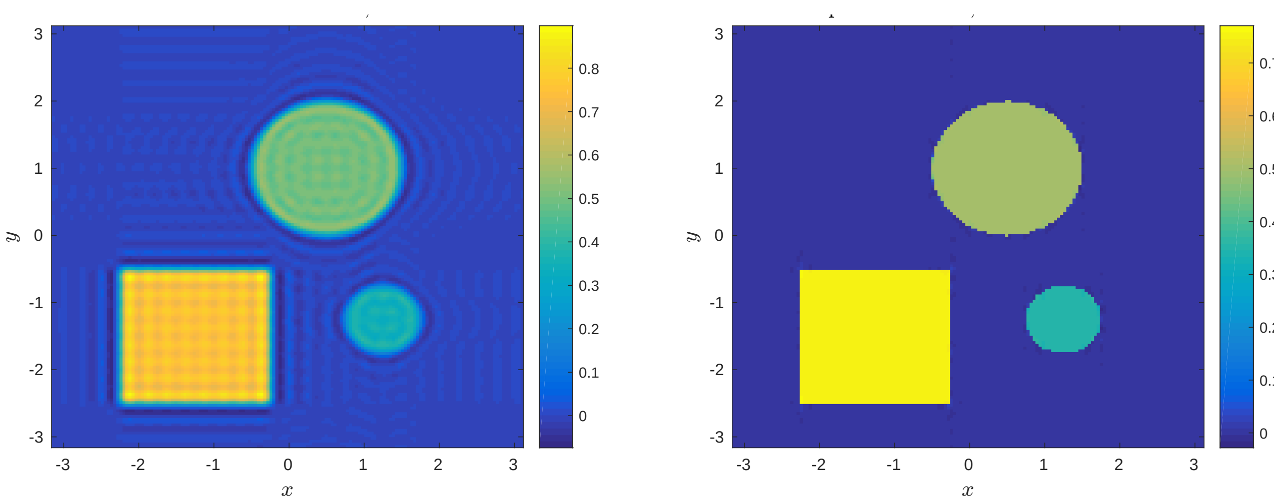
$$\min_{f_2^{CS}} \left\| \left(\mathcal{F} f_2^{CS} - \hat{f}_2 \right)\Big|_{\mathcal{K}} \right\|_2 + \lambda \| f_2^{CS} \|_{TV}. \tag{10}$$

Here, $\mathcal{F}$ denotes the DFT matrix, $\mathcal{K}$ denotes a uniformly randomly chosen set of Fourier frequencies with $|\mathcal{K}| = 2N + 1 = 101$, and λ is a regularization parameter. From the figure, we observe that f_2^{CS} has large errors in the near vicinity of edges. This is due to the inherently discrete nature of compressed sensing methods. If we were given DFT measurements instead of $\hat{f}_2$, the solution of (10) would be exact.

We conclude by presenting preliminary results demonstrating the extension of the proposed method to the two-dimensional case. Fig. 4 plots the reconstruction of the function

$$f_3(x, y) = 0.75\ \mathbf{1}_{[-9/4,-1/4]\times[-5/2,-1/2]} \quad + \quad 0.50\ \mathbf{1}_{\{(x,y)\in\mathbb{R}^2 \ | \ (x-1/2)^2+(y-1)^2\le 1\}} \quad + \\ 0.35\ \mathbf{1}_{\{(x,y)\in\mathbb{R}^2 \ | \ (x-5/4)^2+(y+5/4)^2\le 1/4\}}$$

(here $\mathbf{1}_{\mathcal{A}}$ denotes the indicator function of $\mathcal{A} \subset \mathbb{R}^2$) using its two-dimensional (continuous) Fourier series coefficients $(\widehat{f}_3)_{k,l}$ for $|k|, |l| \le 25$. We can see that the standard Fourier partial sum reconstruction in Fig. 4(a) shows significant Gibbs oscillatory artifacts, while the proposed method in Fig. 4(b) is much more accurate. In essence, the 2D version of the proposed method works by alternately reconstructing (using edge-augmented Fourier sums) along the rows and columns of the image. As with the one-dimensional case, the concentration kernel edge detection method is used to estimate edge locations and heights from the given Fourier data. The interested reader is referred to the companion technical report[15] for additional details.



(a) Standard Fourier Reconstruction, PSNR = 26.97 dB.

(b) Proposed Method, PSNR = 42.76 dB.

Figure 4: 2D Edge-Augmented Fourier Reconstruction using Fourier coefficients $(\widehat{f_3})_{k,l}$ for $|k|, |l| \leq 25$ (Reconstruction error expressed as peak signal to noise ratio (PSNR) in dB)

ACKNOWLEDGMENTS

This work was supported in part by NSF DMS-1416752, NSA grant H98230-16-1-0031, NSF grant DMS-1559776, and was performed as part of the 2016 SURIEM program at Michigan State University.

REFERENCES

[1] Hesthaven, J. S., Gottlieb, S., and Gottlieb, D., [*Spectral Methods for Time-Dependent Problems*], vol. 21 of *Cambridge Monographs on Applied and Computational Mathematics*, Cambridge University Press (2007).

[2] Nishimura, D. G., [*Principles of Magnetic Resonance Imaging*], ch. Imaging Principles, 67–106, Lulu, Raleigh, NC, 1.2 ed. (2010).

[3] Archibald, R. and Gelb, A., "A method to reduce the Gibbs ringing artifact in MRI scans while keeping tissue boundary integrity," *IEEE Trans. Med. Imag.* **21**, 305–319 (Apr. 2002).

[4] Candes, E. J., Romberg, J. K., and Tao, T., "Stable signal recovery from incomplete and inaccurate measurements," *Comm. Pure Appl. Math.* **59**, 1207–1223 (Mar. 2006).

[5] Donoho, D. L., "Compressed sensing," *IEEE Trans. Inf. Theory* **52**, 1289–1306 (Apr 2006).

[6] Lustig, M., Donoho, D., and Pauly, J. M., "Sparse MRI: The application of compressed sensing for rapid MR imaging," *Magn. Reson. Med.* **58**, 1182–1195 (Dec. 2007).

[7] Lustig, M., Donoho, D. L., Santos, J. M., and Pauly, J. M., "Compressed sensing MRI," *IEEE Signal Process. Mag.* **25**, 72–82 (Mar. 2008).

[8] Batenkov, D., "Complete algebraic reconstruction of piecewise-smooth functions from Fourier data," *Math. Comp.* **84**, 2329–2350 (Feb. 2015).

[9] Ongie, G. and Jacob, M., "Recovery of piecewise smooth images from few fourier samples," in [*Int. Conf. Sampling Theory and Applications (SampTA)*], 543–547, IEEE (2015).

[10] Ongie, G. and Jacob, M., "Super-resolution MRI using finite rate of innovation curves," in [*12th Int. Symp. Biomedical Imaging (ISBI)*], 1248–1251, IEEE (2015).

[11] Vetterli, M., Marziliano, P., and Blu, T., "Sampling signals with finite rate of innovation," *IEEE Trans. Signal Process.* **50**(6), 1417–1428 (2002).

[12] Maravic, I. and Vetterli, M., "Sampling and reconstruction of signals with finite rate of innovation in the presence of noise," *IEEE Trans. Signal Process.* **53**(8), 2788–2805 (2005).

[13] Haacke, E. M., Liang, Z.-P., and Izen, S. H., "Superresolution reconstruction through object modeling and parameter estimation," *IEEE Trans. Acoust., Speech, Signal Process.* **37**, 592–595 (Apr. 1989).

[14] Haacke, E. M., Liang, Z.-P., and Izen, S. H., "Constrained reconstruction: A superresolution, optimal signal-to-noise alternative to the Fourier transform in magnetic resonance imaging," *Med. Phys.* **16**(3), 388–397 (1989).

[15] Larriva-Latt, J. et al., "Technical Report: Improved Fourier reconstruction using jump information with applications to MRI," tech. rep., arXiv Preprint 1610.03764, https://arxiv.org/abs/1610.03764 (October 2016).

[16] Potts, D. and Tasche, M., "Parameter estimation for exponential sums by approximate Prony method," *Signal Process.* **90**, 1631–1642 (May 2010).

[17] de Prony, B. G. R., "Essai éxperimental et analytique: sur les lois de la dilatabilité de fluides élastique et sur celles de la force expansive de la vapeur de lalkool,a différentes températures," *J. de lÉcole polytechnique* **1**(22), 24–76 (1795).

[18] Gelb, A. and Tadmor, E., "Detection of edges in spectral data," *Appl. Comput. Harmon. Anal.* **7**, 101–135 (Jul. 1999).

[19] Gelb, A. and Tadmor, E., "Detection of edges in spectral data II. Nonlinear enhancement," *SIAM J. Numer. Anal.* **38**, 1389–1408 (May 2000).

[20] Gelb, A. and Tadmor, E., "Adaptive edge detectors for piecewise smooth data based on the minmod limiter," *J. Sci. Comput.* **28**, 279–306 (Mar. 2006).

[21] Viswanathan, A., Cochran, D., Gelb, A., and Cates, D. M., "Detection of signal discontinuities from noisy Fourier data," in [*Proc. IEEE Asilomar Conf. Signals, Syst. Comput.*], 1705–1708 (Oct. 2008).

[22] Larriva-Latt, J. et al., "FourierRecon: Matlab software for reconstruction from Fourier spectral data," (Sept. 2016).