Math 73/103: Homework on the Cauchy-Riemann Equations

- Problems 1–6 are due Monday, November 21, 2011.
- Problems 7–11 are just for your edification. (But just might appear on some sort of final exam.)
- The remaining problems are due the last day of class, Monday, November 30, 2011.

1. Suppose that Ω is a *region* in $\mathbf{C}$, and that $f \in H(\Omega)$. Show that if $f'(z) = 0$ for all $z \in \Omega$, then f is constant.

Let Ω be a domain in $\mathbf{C}$ and assume that $f : \Omega \to \mathbf{C}$ is a function. Of course, we can view Ω as an open subset of $\mathbf{R}^2$ and define $u, v : \Omega \to \mathbf{R}$ by

$$u(x,y) := \operatorname{Re}\bigl(f(x+iy)\bigr) \quad \text{and} \quad v(x,y) = \operatorname{Im}\bigl(f(x+iy)\bigr)$$

We say that the *Cauchy-Riemann Equations hold at* $z_0 = x_0 + iy_0$ if the partial derivatives of u and v exist at (x_0, y_0) and

$$u_x(x_0,y_0) = v_y(x_0,y_0) \quad \text{and} \quad u_y(x_0,y_0) = -v_x(x_0,y_0). \tag{CR}$$

We often abuse notation slightly, and say that (CR) amounts to $f_y(z_0) = if_x(z_0)$. (Just to be specific, $f_x(x_0 + iy_0) := u_x(x_0,y_0) + iv_x(x_0,y_0)$.)

2. Suppose that $f'(z_0)$ exists. Show that

$$f_x(z_0) = f'(z_0) = -if_y(z_0). \tag{1}$$

Conclude that the Cauchy-Riemann equations hold at z_0 whenever $f'(z_0)$ exists. Verify (1) when $f(z) = z^2$.

3. Suppose that Ω is a region and $f \in H(\Omega)$. Show that if f is real-valued in Ω, then f is constant.

4. Suppose that Ω is a region and $f \in H(\Omega)$. Suppose that $z \mapsto |f(z)|$ is constant on Ω. Show that f must be constant. (Consider $|f(z)|^2$.)

We let f, u, v and Ω be as above. Define

$$F : \Omega \subset \mathbf{R}^2 \to \mathbf{R}^2 \quad \text{by} \quad F(x,y) = \big(u(x,y), v(x,y)\big).$$

Pretend that you remember that F is differentiable at $(x_0, y_0) \in \Omega$ if there is a linear function $L : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{\|F(x_0+h, y_0+k) - F(x_0,y_0) - L(h,k)\|}{\|(h,k)\|} = 0,$$

in which case, the partials of u and v must exist and L is given by the Jacobian Matrix

$$[L] = \begin{pmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{pmatrix}.$$

(Of course, here $\|(x,y)\| = \sqrt{x^2+y^2} = |x+iy|$.)

5. Let f, F, u, v and Ω be as above. Let $z_0 = x_0 + iy_0 \in \Omega$. Show that $f'(z_0)$ exists if and only if the Cauchy-Riemann equations hold at z_0 and F is differentiable at (x_0, y_0). (Hint: if we let $z = h + ik$ and if T is given by the matrix

$$[T] = \begin{pmatrix} u_x(x_0,y_0) & -v_x(x_0,y_0) \\ v_x(x_0,y_0) & u_x(x_0,y_0) \end{pmatrix},$$

then

$$\|F(x_0+h, y_0+k) - F(x_0,y_0) - T(h,k)\| = |f(z+z_0) - f(z_0) - \omega z|,$$

where $\omega = u_x(x_0,y_0) + iv_x(x_0,y_0) = f_x(z_0)$. Then remember (1).)

Problem #5 has an important Corollary. We learn in multivariable calculus, that F is differentiable at (x_0, y_0) if the partial derivatives of u and v exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0). Hence we get as a Corollary of problem #5, with f, u and v defined as above, that if u and v have continuous partial derivatives in a neighborhood of (x_0, y_0) and if the Cauchy-Riemann equations hold at z_0, then $f'(z_0)$ exists. Use this observation in problem #6.

6. Define $\exp : \mathbf{C} \to \mathbf{C}$ by $\exp(x+iy) = e^x\big(\cos(y) + i\sin(y)\big)$. Show that $\exp \in H(\mathbf{C})$ and $\exp'(z) = \exp(z)$ for all $z \in \mathbf{C}$.

If Ω is open in $\mathbf{C}$ or $\mathbf{R}^2$, then we say $u : \Omega \to \mathbf{R}$ is *harmonic* if it has continuous second partial derivatives and if it is a solution to Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{L}$$

7. Suppose that $f \in H(\Omega)$. Let $u(x, y) = \operatorname{Re}\bigl(f(x + iy)\bigr)$. Show that u is harmonic in Ω.

8. Suppose that $u : \Omega \to \mathbf{R}$ is harmonic. We say $v : \Omega \to \mathbf{R}$ is a *harmonic conjugate* for u if $f(x + iy) = u(x, y) + iv(x, y)$ defines a holomorphic function on Ω. Find all harmonic conjugates for $u(x, y) = 2xy$.

For the purposes of this assignment only, we'll call a region Ω a *SC region* if every $f \in H(\Omega)$ has an antiderivative in Ω. For example, we have shown in lecture that every convex region is a SC region. Later, I hope that we'll see that any simply connected region is a SC region. In fact, a region is a SC-region if and only if it is simply connected.

9. Suppose that Ω is a SC region and that u is harmonic in Ω. Show that u has a harmonic conjugate in Ω. (Hint: we need to find a function $f \in H(\Omega)$ such that $u = \operatorname{Re}(f)$. However, let $g = u_x - iu_y$. Show that $g \in H(\Omega)$ and consider an anti-derivative f for g in Ω. You may use without proof the fact that if $w : \Omega \to \mathbf{R}$ is continuous and $w_x \equiv 0 \equiv w_y$ in Ω, then w is constant.)

If $u = \operatorname{Re}(f)$, then $u_x = \operatorname{Re}(f')$ and $u_y = \operatorname{Re}(-if')$. Thus, it is a consequence of question #9 that every harmonic function has continuous partial derivatives of all orders.

10. Just as in question #6, we'll be fancy and write $\exp(z)$ in place of e^z. Suppose that Ω is a SC region and that $0 \notin \Omega$. Then show there is a $f \in H(\Omega)$ such that

$$\exp\bigl(f(z)\bigr) = z.$$

We call f a *branch of* $\log(z)$ *in* Ω. (Hint: start by letting f be an antiderivative of $1/z$. and recall that $\exp(z) = a$ has infinitely many solutions for all $a \neq 0$.)

11. Show that $f(z) = 1/z$ can't have an antiderivative in the punctured complex plane $\mathbf{C}^* := \mathbf{C} \setminus \{0\}$. Conclude that there is no (holomorphic) branch of $\log z$ in $\mathbf{C}^*$.

12. Work problems 3, 4, 5, 13 and 20 on pages 227–230 of your text (at the end of Chapter 10.)

HINT FOR #4: Estimate $|f^{k+1}(z)|$.

HINT FOR #5: The hypotheses of the problem don't allow us to conclude even that the limit function $f = \lim_n f_n$ is continuous. Instead, you'll have to prove that $\{ f_n \}$ is uniformly Cauchy. You may want to use (and prove) that if $g_n(z) \le M$ for all $z \in \gamma^*$ and g_n converges pointwise to a 0, then

$$\int_\gamma g(z)\, dz \to 0.$$

HINT FOR #20: Notice that $f_n' \to f'$ uniformly on compact subsets of Ω. Show this implies $f_n'/f_n \to f'/f$ uniformly on any γ^* provided $f \neq 0$ on γ^*.