HOMEWORK #9

NOT TO BE TURNED IN

Problem 1. Find all ideals of $\mathbb{Q}[x]$ containing $(x^4 - 1)$. For each ideal you find, determine whether it is maximal, prime, both, or neither.

Problem 2 (D&F 9.2.4). Let F be a finite field. Prove that $F[x]$ contains infinitely many primes.

Problem 3 (D&F 9.4.1,2). Determine whether the following polynomials are irreducible in the rings indicated. For those that are reducible, determine their factorization into irreducibles. Let $\mathbb{F}_p$ denote the finite field $\mathbb{Z}/p\mathbb{Z}$:

(a) $x^2 + x + 1$ in $\mathbb{F}_2[x]$.
(b) $x^4 - 4x^3 + 6$ in $\mathbb{Z}[x]$.
(c) $x^6 + 30x^5 - 15x^3 + 6x - 120$ in $\mathbb{Z}[x]$.
(d) $x^4 + 1$ in $\mathbb{F}_5[x]$.
(e) $x^3 + x + 1$ in $\mathbb{F}_3[x]$.
(f) $x^4 + 4x^3 + 6x^2 + 2x + 1$ in $\mathbb{Z}[x]$.

Problem 4 (D&F 9.4.3). Show that the polynomial $(x-1)(x-2)\cdots(x-n) - 1$ is irreducible in $\mathbb{Z}[x]$ for all $n \geq 1$.
[Hint: Note that in a UFD, the fundamental theorem of arithmetic holds.]

Problem 5 (D&F 9.4.6). Construct a field of order 9. You may exhibit this field as $F[x]/(f(x))$ for some F and f. You may use D&F problem 9.2.2.

Problem 6 (D&F 9.4.11). Prove that $x^2 + y^2 - 1$ is irreducible in $\mathbb{Q}[x, y]$.

Problem 7 (D&F 9.4.14). Factor the polynomials $x^8 - 1$ and $x^6 - 1$ into irreducibles in each of the following rings:

(a) $\mathbb{Z}[x]$
(b) $\mathbb{Z}/2\mathbb{Z}[x]$
(c) $\mathbb{Z}/3\mathbb{Z}[x]$