HOMEWORK #8

DUE 11/13/13 AT START OF CLASS

Problem 1 (D&F 8.1.3). Let R be a Euclidean Domain. Let m be the minimum integer in the set of norms of nonzero elements of R. Prove that every nonzero element of norm m is a unit. Deduce that a nonzero element of norm zero (if such an element exists) is a unit.

Problem 2 (D&F 8.1.7). Find a generator for the ideal $(85, 1+13i)$ in $\mathbb{Z}[i] = \mathbb{Z}[\sqrt{(-1)}]$.
[Hint: $\mathbb{Z}[i]$ is a Euclidean Domain with respect to the norm $N(a+bi) = a^2 + b^2$ (cf page 272).]

Problem 3 (D&F 8.1.11,8.2.2). Let R be a commutative ring with 1 and let a and b be nonzero elements of R. A *least common multiple* of a and b is an element e of R such that

(i) $a \mid e$ and $b \mid e$, and
(ii) if $a \mid e'$ and $b \mid e'$ then $e \mid e'$

(a) Prove that a least common multiple of a and b (if such exists) is a generator for the unique largest principal ideal contained in $(a) \cap (b)$.
(b) Deduce that any two nonzero elements in a Euclidean Domain have a least common multiple which is unique up to multiplication by a unit.
(c) Prove that in a Euclidean Domain the least common multiple of a and b is $\frac{ab}{(a,b)}$, where (a, b) is the greatest common divisor of a and b.
(d) Prove that any two nonzero elements of a PID have a least common multiple.

Problem 4 (D&F 8.2.3). Let R be an integral domain and suppose that every prime ideal in R is principal. This exercise proves that every ideal of R is principal, i.e. R is a PID.

First let $\mathcal{I}$ be the set of ideals of R that are not principal. Assume that if $\mathcal{I}$ is nonempty then it has a maximal element under inclusion. (Challenge: read about Zorn's lemma and prove this statement).

(a) Let I be a maximal element of $\mathcal{I}$ and let $a, b \in R$ with $ab \in I$ but $a \notin I$ and $b \notin I$. Let $I_a = (I, a)$ be the ideal generated by I and a. Let $I_b = (I, b)$ be the ideal generated by I and b, and define $J = \{r \in R \mid rI_a \subseteq I\}$. Prove that $I_a = (\alpha)$ and $J = (\beta)$ are principal ideals in R with $I \subsetneq I_b \subseteq J$ and $I_a J = (\alpha\beta) \subseteq I$..

(b) If $x \in I$ show that $x = s\alpha$ for some $s \in J$. Deduce that $I = I_a J$ is principal, a contradiction, and conclude that R is a PID.

Problem 5. In each of the following rings, give at least two distinct factorizations of some nonzero non-unit into irreducibles.

(a) $\mathbb{Q}[x^2, x^3]$. [This ring consists of finite sums of elements of the form $a(x^2)^i(x^3)^j$ (see pages 296-297 in D&F).]

(b) $\mathbb{Z}[\sqrt{-13}]$.

Problem 6 (D&F 9.2.3). Let F be a field and let $f(x)$ be a polynomial in $F[x]$. Prove that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible.

Problem 7 (D&F 9.2.6). Describe the ring structure of the following rings:

(a) $\mathbb{Z}[x]/(2)$.

(b) $\mathbb{Z}[x]/(x)$.

(c) $\mathbb{Z}[x]/(x^2)$.

Challenge Problem. Determine the characteristics (cf Homework 7 #4) of each of the rings in Problem 7.