HOMEWORK #7

DUE 11/06/13 AT START OF CLASS

Problem 1. Find all ring homomorphisms $\mathbb{Z} \to \mathbb{Z}$.

Problem 2 (D&F 7.3.10). Determine which of the following are ideals of the ring $\mathbb{Z}[x]$. Justify your answers.

(a) the set of all polynomials whose constant term is a multiple of 3.
(b) the set of all polynomials whose coefficient of x^2 is a multiple of 3.
(c) the set of all polynomials in which only even powers of x appear.
(d) the set of all polynomials whose constant term, coefficient of x, and coefficient of x^2 are 0.
(e) the set of polynomials whose coefficients sum to zero
(f) the set of polynomials $p(x)$ such that $p'(0) = 0$, where $p'(x)$ is the usual first derivative of $p(x)$ with respect to x.

Problem 3 (D&F 7.3.17). Let R and S be nonzero rings with identity. Let $\phi : R \to S$ be a nonzero homomorphism of rings.

(a) Prove that if $\phi(1_R) \neq 1_S$ then $\phi(1_R)$ is a zero divisor in S. Deduce that if S is an integral domain then every nonzero ring homomorphism from R to S sends the identity of R to the identity of S.
(b) Prove that if $\phi(1_R) = 1_S$ then $\phi(u)$ is a unit in S and $\phi(u^{-1}) = \phi(u)^{-1}$ for each unit u of R.

Problem 4 (D&F 7.3.26,28). The *characteristic* of a ring R is the smallest postive integer n such that $1 + 1 + \cdots + 1 = 0$ (n times) in R; if no such integer exists the characteristic of R is said to be 0.

(a) Determine the characteristic of the rings $\mathbb{Q}, \mathbb{Z}[x], \mathbb{Z}/n\mathbb{Z}[x]$.
(b) Prove that if p is a prime and R is a commutative ring of characteristic p then $(a + b)^p = a^p + b^p$ for all $a, b \in R$. (you may use D&F 7.3.25).
(c) Prove that an integral domain has characteristic p, where p is either a prime or 0.

Problem 5 (D&F 7.3.34). Let I and J be ideals of R. For the definition of $I + J$ and IJ, see page 247 of D&F.

(a) Prove that $I + J$ is the smallest ideal of R containing both I and J.
(b) Prove that IJ is an ideal contained in $I \cap J$.
(c) Give an example where $IJ \neq I \cap J$.
(d) Prove that if R is commutative with identity and $I + J = R$ then $IJ = I \cap J$.

Problem 6 (D&F 7.4.8). Let R be an integral domain. Prove that $(a) = (b)$ for some elements $a, b \in R$ if and only if $a = ub$ for some unit u of R.

Problem 7 (D&F 7.4.17). You may want to read and work through problem 7.4.14 before considering this problem. Let $x^3 - 2x + 1$ be an element of the polynomial ring $\mathbb{Z}[x]$ and use the bar notation to denote passage to the quotient ring $\mathbb{Z}[x]/(x^3 - 2x + 1)$. Let $p(x) = 2x^7 - 7x^5 + 4x^3 - 9x + 1$.

(a) Express $p(x)$ in the form $\overline{f(x)}$ for some polynomial $f(x)$ of degree ≤ 2.
(b) Prove that $\mathbb{Z}[x]/(x^3 - 2x + 1)$ is not an integral domain.

Problem 8 (D&F 7.4.19). Let R be a finite commutative ring with identity. Prove that every prime ideal of R is a maximal ideal.

Challenge Problem. Find an ideal I such that $\mathbb{R}[x]/I \cong \mathbb{C}$. Under this isomorphism, what does i correspond to?