HOMEWORK #6

DUE 10/30/13 AT START OF CLASS

Problem 1 (D&F 4.5.30). How many elements of order 7 must there be in a simple group of order 168?

Problem 2. Let G be a group of order 48. Use the following hints to show that G has a normal subgroup of order 8 or 16:

(a) If there is more than one Sylow 2-subgroup, let H and K be any two (distinct) Sylow 2-subgroups. Show that $|H \cap K| = 8$.
(b) Show that $H, K \subseteq N_G(H \cap K)$.
(c) Show that $G = N_G(H \cap K)$.

Problem 3. Let G be a group of order 231 with only one Sylow 3-subgroup. Show that G is cyclic.

Problem 4 (D&F 5.2.2,3). Give the lists of invariant factors for all abelian groups of the specified orders. Also, give the lists of elementary divisors for all abelian groups of the specified order and then match each list with the corresponding list of invariant factors.

(a) order 270
(b) order 9801

Problem 5. Let G be a finite abelian group of order n. Prove that if k divides n, then G has a subgroup of order k.

For problems 6-9, let R be a ring with 1.

Problem 6 (D&F 7.1.1). Show that $(-1)^2 = 1$ in R.

Problem 7 (D&F 7.1.7). The *center* of a ring R is

$$\{z \in R \mid zr = rz \text{ for all } r \in R\}.$$

Prove that the center of a ring is a subring that contains the identity. Prove that the center of a division ring is a field.

Problem 8 (D&F 7.1.11). Prove that if R is an integral domain and $x^2 = 1$ for some $x \in R$ then $x = \pm 1$.

Problem 9 (D&F 7.1.12). Prove that any subring of a field which contains the identity is an integral domain.