HOMEWORK #5

DUE 10/23/13 AT START OF CLASS

Problem 1 (D&F 3.3.3). Let G be a group, H a normal subgroup of G with $[G : H] = p$ for some prime p. Prove for all subgroups $K \leq G$ that either

(i) $K \leq H$, or
(ii) $G = HK$ and $|K : K \cap H| = p$.

Problem 2. Let n and m be positive integers. Let $[n, m]$ denote the least common multiple of n and m

(a) Use the First Isomorphism Theorem to show that there is an injective homomorphism of $\mathbb{Z}/[n, m]\mathbb{Z}$ into the direct product $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.
(b) Show that your map in part (a) is surjective (and therefore an isomorphism) if and only if $(n, m) = 1$. Conclude that in this case $\mathbb{Z}/nm\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.

Problem 3. Let $\mathbb{Z}[x]$ denote the set of polynomials with coefficients in $\mathbb{Z}$. It is clear that $\mathbb{Z}[x]$ is an additive group.

(a) Define $\phi : \mathbb{Z}[x] \to \mathbb{Z}$ by $\phi(f) = f(0)$. Show that ϕ is a surjective homomorphism and compute its kernel. What statement does the first isomorphism theorem yield?
(b) Do the same as above with $\phi : \mathbb{Z}[x] \to \mathbb{Z}$ defined by $\phi(f) = f(3)$.

Problem 4 (D&F 3.4.2). Exhibit all three composition series for the quaternions Q_8. List the composition factors in each case.

Problem 5 (D&F 3.4.12). A group G is *solvable* if there is a chain of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_s = G$$

such that G_i/G_{i-1} is abelian for $i = 1, \ldots, s$. Prove, without using the Feit-Thompson Theorem, that the following are equivalent:

(i) every group of odd order is solvable.
(ii) the only simple groups of odd order are those of prime order.

Problem 6 (D&F 3.5.3,4).

(a) Prove that S_n is generated by $\{(i\ i+1) \mid 1 \leq i \leq n-1\}$.
(b) Prove that $S_n = \langle (12), (12\ldots n) \rangle$ for $n \geq 2$.

Challenge Problem. Prove that the 15-puzzle arrangement given in class is unsolvable.

Problem 7 (D&F 1.7.5). For a group action of G on A, the *kernel of the action* is the set $\{g \in G \mid g.a = a\ \forall a \in A\}$. Prove that the kernel of the action of the group G on the set A is the same as the kernel of the corresponding permutation representation $\phi : G \to S_A$.

Problem 8 (D&F 4.2.8). Prove that if H has finite index n in G then there is a normal subgroup K of G with $K \leq H$ and $[G : K] \leq n!$.

Problem 9 (D&F 4.2.10). Prove that every non-abelian group of order 6 has a non normal subgroup of order 2. Use this to classify all groups of order 6 (produce an injective homomorphism into S_3).

Problem 10 (D&F 4.3.5). If the center of G is of index n, prove that every conjugacy class has at most n elements.

Problem 11 (D&F 4.3.6). Assume that G is a non-abelian group of order 15. Prove that $Z(G) = 1$. Use the fact that $\langle g \rangle \leq C_G(g)$ for all $g \in G$ to show there is at most one possible class equation for G.

Problem 12 (D&F 4.3.7). For $n = 6$ make a list of the partitions of n and give representatives for the corresponding conjugacy classes of S_n (read pgs 125-128).