HOMEWORK #4

DUE 10/16/13 AT START OF CLASS

Problem 1. Let G be a group and let H be a subgroup of G.

(a) Let $\phi : G \to H$ be a homomorphism. Prove that $\ker(\phi) \trianglelefteq G$.
(b) Prove that $N_G(H) \leq G$ and $H \leq N_G(H)$.

Problem 2 (D&F 3.1.5). Prove that the order of the element gH in the quotient group G/H is n, for n the smallest positive integer such that $g^n \in H$ (and gH has infinite order if no such positive integer exists). Give an example to show that the order of gH in G/H may be strictly smaller than the order of g in G.
[Hint: you may use without proof that $(gH)^a = g^a H$ for all $a \in \mathbb{Z}$.]

Problem 3 (D&F 3.1.22). Let G be a group.

(a) Prove that if H and K are normal subgroups of G then their intersection is also a normal subgroup of G.
(b) Prove that the intersection of an arbitrary nonempty collection of normal subgroups of a group is a normal subgroup (do not assume the collection is countable).

Problem 4 (D&F 3.1.42). Let H and K be normal subgroups of G with $H \cap K = 1$. Prove that $xy = yx$ for all $x \in H$ and $y \in K$.
[Hint: show $x^{-1}y^{-1}xy \in H \cap K$.]

Problem 5 (D&F 3.2.11). Let $H \leq K \leq G$. Prove that $[G : H] = [G : K][K : H]$ (do not assume G is finite).

Problem 6 D&F 3.2.16. Use Lagrange's theorem in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ to prove *Fermat's Little Theorem*: if p is prime then $a^p \equiv a \pmod p$ for all $a \in \mathbb{Z}$.

Problem 7 (D&F 3.2.19). Prove that if N is a normal subgroup of the finite group G and $(|N|, [G : N]) = 1$ then N is the unique subgroup of G of order $|N|$.

Problem 8 (D&F 3.1.36). Prove that if $G/Z(G)$ is cyclic then G is abelian.

Challenge Problem. Let G be a group of order pq. Prove that either G is abelian or $Z(G) = 1$.