HOMEWORK #3

DUE 10/9/13 AT START OF CLASS

Problem 1 (D&F 2.1.1,2.1.2). In each of the following, determine (and prove) whether or not the specified subset is a subgroup of the given group.

(a) The set of 2-cycles in S_n for $n \geq 3$.
(b) For fixed $n \in \mathbb{Z}^+$, the set of rational numbers whose denominators divide n (under addition).
(c) The set of complex numbers of the form $a + ai$, $a \in \mathbb{R}$ (under addition).
(d) The set of odd integers in $\mathbb{Z}$ together with 0 (under addition).
(e) The set of reflections in D_{2n} for $n \geq 3$.

Problem 2 (D&F 1.6.13,14). Let G and H be groups and let $\phi : G \to H$ be a homomorphism. Recall that the *kernel* of ϕ is the subset $\{g \in G \mid \phi(G) = e_H\} \subseteq G$, while the *image* of ϕ is the subset $\{\phi(g) \mid g \in G\} \subseteq H$. Prove that the kernel of ϕ is a subgroup of G and the image of ϕ is a subgroup of H.

Problem 3 (D&F 2.1.5). Let G be a group with $|G| = n > 2$. Prove that G cannot have a subgroup H with $|H| = n - 1$. Do not use Lagrange's Theorem.

Problem 4 (D&F 2.1.8). Let H and K be subgroups of G. Prove that $H \cup K$ is a subgroup of G if and only if either $H \subseteq K$ or $K \subseteq H$.

Challenge Problem. Let H be a subgroup of the additive group of rational numbers with the property that $\frac{1}{x} \in H$ for every nonzero element x of H. Prove that either $H = 0$ or $H = \mathbb{Q}$.

Problem 5. Find all subgroups of $Z_{32} = \langle x \rangle$, giving a generator for each. Describe the containments between these subgroups.

Problem 6 (D&F 2.3.12). Prove that the following groups are *not* cyclic:

(a) $Z_2 \times Z_2$
(b) $Z_2 \times \mathbb{Z}$
(c) $\mathbb{Z} \times \mathbb{Z}$

Problem 7. Let G be a group with an element $g \neq e$ such that g is an element of every subgroup of G, except the trivial subgroup $\{e\}$. Prove that every element of G has finite order.

Problem 8. Show that Z_n has an element as described in Problem 8 if and only if n is a power of a prime.