HOMEWORK #2

DUE 10/2/13 AT START OF CLASS

Problem 1 (D&F 1.2.2,3). Consider the dihedral group D_{2n} with the usual presentation $D_{2n} = \langle r, s | r^n = s^2 = 1, rs = sr^{-1} \rangle$. Use these generators and relations to show:

(a) If $x \in D_{2n}$ is not a power of r, then $rx = xr^{-1}$.
(b) Every element of D_{2n} which is not a power of r has order 2.

Problem 2 (D&F 1.2.13). Recall that a regular icosahedron is a 20-faced polyhedron, with each face an equilateral triangle. Let G be the group of rigid motions in $\mathbb{R}^3$ (symmetries) of a regular icosahedron. Show that $|G| = 60$.

Problem 3 (D&F 1.3.1). Let σ be the permutation

$$1 \mapsto 3 \quad 2 \mapsto 4 \quad 3 \mapsto 5 \quad 4 \mapsto 2 \quad 5 \mapsto 1$$

and let τ be the permutation

$$1 \mapsto 5 \quad 2 \mapsto 3 \quad 3 \mapsto 2 \quad 4 \mapsto 4 \quad 5 \mapsto 1$$

Find the cycle decompositions of each of the following permutations:

$$\sigma, \tau, \sigma^2, \sigma\tau, \tau\sigma, \tau^2\sigma.$$

Problem 4 (D&F 1.3.10). Prove that if σ is the m-cycle $(a_1\ a_2\ \ldots\ a_m)$ then for all $1 \leq i \leq m$, $\sigma^i(a_k) = a_{k+i}$, where $k + i$ is replaced by its least positive residue mod m. Deduce that $|\sigma| = m$.

Problem 5 (D&F 1.3.9,11).

(a) Let σ be the 10-cycle $(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10)$. For which positive integers i is σ^i also a 10-cycle?
(b) Let τ be the 8-cycle $(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)$. For which positive integers i is τ^i also a 8-cycle?
(c) Let σ be the m-cycle $(1\ 2\ \ldots\ m)$. Using parts (a) and (b), complete the following statement:

σ^i is an m-cycle if and only if...

Challenge: Prove this.

Problem 6 (D&F 1.3.15). Prove that the order of an element in S_n equals the least common multiple of the lengths of the cycles in its cycle decomposition.
[Hint: Use Problem 5 from Homework#1]

Problem 7 (D&F 1.4.4). Show that if n is not prime then $\mathbb{Z}/n\mathbb{Z}$ is not a field.

Problem 8 (D&F 1.6.3). Let G, H be groups. If $\phi : G \to H$ is an isomorphism, prove that G is abelian if and only if H is abelian.

Problem 9 (D&F 1.6.11). Let G, H be groups. Prove that $G \times H \cong H \times G$.

Problem 10 (D&F 1.6.14). Let G, H be groups and $\phi : G \to H$ a homomorphism. Define the *kernel* of ϕ to be

$$\ker(\phi) = \{g \in G |\ \phi(g) = e_H\}.$$

Prove that ϕ is injective if and only if $\ker(\phi) = \{e_G\}$.

Problem 11 (D&F 1.6.17). Let G be a group. Prove that the map from G to itself defined by $g \mapsto g^{-1}$ is a homomorphism if and only if G is abelian.

Challenge Problem. Consider the symmetric group S_n. Show that the two elements $(1\ 2)$ and $(1\ 2\ \dots\ n)$ form a generating set for S_n.