Math 6 – Permutations Worksheet

If $S = \{1, 2, 3, \ldots, n\}$, then the set of permutations $\mathrm{Sym}(S)$ on the set S is often written S_n. The purpose of this worksheet is for you to get some practice computing with the elements of S_n. We write σ_0 for the identity permutation, i.e., the permutation that takes every element to itself.

1. How many elements are there in S_3? List them. How many elements are in S_5? (You don't need to list them.) How about in S_n?

2. Suppose that $n = 5$ and that we are given the following two elements of S_5:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}.$$

   Compute $\sigma\tau$ and $\tau\sigma$. What is σ^{-1}?

3. Find the decomposition into disjoint cycles of each of the following elements of S_6:

   (a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 6 & 1 & 4 \end{pmatrix}$

   (b) $(1\ 2)(1\ 3\ 4\ 2)(4\ 3\ 5)(2\ 5)$

   (c) $(1\ 2\ 3)^{-1}(1\ 4\ 5\ 3\ 6)(1\ 2\ 3)$

4. Let σ, τ be the permutations from problem 2. Express σ as a product of disjoint cycles. Express τ as a product of disjoint cycles. Express $\sigma\tau$ as a product of disjoint cycles.

5. Now let $n = 6$. Suppose $\sigma = (1\ 2\ 3)$. What is σ^{-1}? What is σ^2? What is σ^3? In the sequence $\sigma, \sigma^2, \sigma^3, \ldots,$ which terms are the identity permutation?

6. Again let $n = 6$. Let $\sigma = (1\ 2\ 3)$ and $\tau = (4\ 5)$. Which terms of the sequence $\tau, \tau^2, \tau^3, \ldots$ are the identity permutation of S_6? Let $\phi = \sigma\tau = (1\ 2\ 3)(4\ 5)$. Which terms of the sequence $\phi, \phi^2, \phi^3, \ldots$ are the identity permutation?

7. A two-element cycle is called a *transposition*. For example, $(1\ 3)$ is the transposition that switches 1 and 3 and leaves every other element fixed. Suppose $n = 6$ and consider the cycle $(1\ 2\ 3)$. Can you figure out a way of writing $(1\ 2\ 3)$ as a product of transpositions? What about $(1\ 2\ 3\ 4)$? Can you write the permutation τ of problem 2 as a product of transpositions?

8. Can every cycle be written as a product of transpositions? What about every permutation?

9. Marty the Magician uses a 10-card deck in all of his card tricks. He shuffles the deck by placing the top card at the bottom, cutting the deck in half and then placing the bottom half on top of the top half. How many times must this shuffle be repeated to get the cards back in the original order?

10. Let $n = 6$. There are a number of ways of writing σ_0 as a product of transpositions. For example, we can write $\sigma_0 = (1\ 2)(1\ 2)$, or $\sigma_0 = (1\ 2)(1\ 3)(1\ 2)(1\ 3)(1\ 2)(1\ 3)$. In both cases, the identity is written as the product of an *even* number of transpositions. Can you find a way of writing σ_0 as the product of an *odd* number of transpositions?