Math 6 Midterm Review Solutions

1. (a) statement
   (b) not a statement—truth or falsity depends on the reference for the pronoun "she"
   (c) statement
   (d) not a statement—truth or falsity depends on the reference for the variable x
   (e) not a statement—neither true nor false

2. (a) $p \wedge q \wedge r$
   (b) $p \wedge \sim q$
   (c) $p \wedge (\sim q \vee \sim r)$
   (d) $\sim p \wedge q \wedge \sim r$
   (e) $\sim p \vee (q \wedge r)$

3. (a) Hal isn't a math major or Hal's sister isn't a computer science major.
   (b) The connector isn't loose and the machine isn't unplugged.
   (c) Today is Thanksgiving and tomorrow isn't Friday.
   (d) The number n is prime, but n isn't odd and n isn't 2.
   (e) Tom is Ann's father, but Jim isn't her uncle or Sue isn't her aunt.
   (f) $\exists$ a computer program P such that P is correct and P doesn't compile without error messages.
   (g) $\exists$ integers a, b, and c such that $a - b$ is even and $b - c$ is even, but $a - c$ isn't even.
   (h) $\exists$ an animal x such that x is a cat, but x doesn't have whiskers or x doesn't have claws.

4. (a) If this loop does not contain a "stop" or a "go to," then this loop will repeat exactly N times.
   (b) If I catch the 8:05 bus, then I will be on time for work.
   (c) If a person graduates with honors, then the person has a grade-point average of at least 3.7.

5. false—the negation of an implication is not an implication

6. (a) If tomorrow isn't Friday, then today isn't Thanksgiving.
   (b) If n isn't odd and n isn't 2, then n isn't prime.
   (c) If Jim isn't Ann's uncle or Sue isn't Ann's aunt, then Tom isn't Ann's father.
   (d) $\forall$ computer programs P, if P doesn't compile without error messages, then P isn't correct.
   (e) $\forall$ animals A, if A doesn't have whiskers or A doesn't have claws, then A isn't a cat.

7. (a) $\sqrt{2}$ isn't rational.
   (b) This is a "while" loop.
   (c) Logic isn't easy.
   (d) This polygon isn't a triangle.

(e) They didn't telephone.

8. not valid—Jules could have solved the problem incorrectly but still have obtained the answer 2

9. Let h be the statement "This house is next to a lake," k the statement "The treasure is in the kitchen," e the statement "The tree in the front yard is an elm," f the statement "The treasure is buried under the flagpole," b the statement "The tree in the backyard is an oak," and g the statement "The treasure is in the garage." Then we have the following argument.

1. h	hypothesis
2. $h \to \sim k$	hypothesis
3. $\sim k$	modus ponens (1, 2)
4. $e \to k$	hypothesis
5. $\sim e$	modus tollens (3, 4)
6. $e \vee f$	hypothesis
7. f	disjunctive syllogism (5, 6)

The treasure is buried under the flagpole.

10. (a) false
    (b) true
    (c) false
    (d) true
    (e) false
    (f) true

11. (a), (e), (f)

12. (a) $\forall$ dinosaurs x, x is extinct.
    (b) $\forall$ real numbers x, x is positive, negative, or zero.
    (c) $\forall$ irrational numbers x, x is not an integer.
    (d) $\forall$ logicians x, x isn't lazy.

13. (a) $\exists$ an exercise x such that x has an answer.
    (b) $\exists$ a real number x such that x is rational.

14. (b), (d)

15. (a) *correct negation*: There are two irrational numbers whose sum is rational (a number is either rational or irrational, but not both).
    (b) *correct negation*: There is an integer n such that n^2 is even and n isn't even.

16. (a) i. There are animals of every color.
    ii. $\exists$ a color C such that $\forall$ animals A, A isn't colored C.
    For some color, there is no animal of that color.
    (b) i. There is a book that everyone has read.
    ii. $\forall$ books b, $\exists$ a person p such that p hasn't read b.
    For all books, there is a person who hasn't read it.

17. $\forall$ people x, $\exists$ a person y such that x is older than y.

18. $\exists$ a person x such that $\forall$ people y, x is older than y.

19. (a) i. $\forall$ people x, $\exists$ a person y such that x trusts y.
    ii. $\exists$ a person x such that $\forall$ people y, x doesn't trust y.
    There is a person who doesn't trust anybody.

    (b) i. $\exists$ a person x such that $\forall$ people y, x trusts y.
    ii. $\forall$ people x, $\exists$ a person y such that x doesn't trust y.
    Everybody doesn't trust somebody.

    (c) i. $\forall$ even integers x, $\exists$ an integer y such that $x = 2y$.
    ii. $\exists$ an even integer x such that $\forall$ integers y, $x \neq 2y$.
    There is an even integer that doesn't equal twice any other integer.

    (d) i. $\exists$ a program x such that $\forall$ questions y posed to x, x gives the correct answer to y.
    ii. $\forall$ programs x, $\exists$ a question y posed to x such that x doesn't give the correct answer to y.
    For all programs, there is question that can be posed to it that it doesn't answer correctly.

20. Let C be the set of all workers who are college graduates and D the set of all workers who belong to a union. Then $n(C') = 60$, $n(D) = 30$, and $n(D' \cap C) = 20$. Note that $n(C \cup D) = n(D' \cap C) + n(D) = 20 + 30 = 50$, so $n((C \cup D)') = n(U) - n(C \cup D) = 100 - 50 = \underline{\underline{50}}$.

21. Let V be the set of all lines with a verb and A the set of all lines with an adjective. Then $n(V) = 11$, $n(A) = 9$, and $n(V \cap A) = 7$. It follows that $n(V \cap A') = n(V) - n(V \cap A) = 11 - 7 = \underline{\underline{4}}$ and $n(A \cap V') = n(A) - n(A \cap V) = 9 - 7 = \underline{\underline{2}}$. Finally, by the inclusion-exclusion principle, $n(V \cup A) = n(V) + n(A) - n(V \cap A) = 11 + 9 - 7 = 13$, so $n((V \cup A)') = n(U) - n(V \cup A) = 14 - 13 = \underline{\underline{1}}$.

22. Note that there are $\binom{2}{1} \cdot \binom{3}{1} \cdot \binom{2}{1} = 12$ ways to select a temperature, pressure, and level of catalyst for a run; hence, if the chemical engineer wanted to observe each temperature-pressure-catalyst combination once, she would have to do 12 different runs. To observe each combination exactly twice, she therefore needs to do $\underline{\underline{24}}$ different runs.

23. (a) For each selection, there are 20 different options, so there are $\underline{\underline{20^3}}$ different combinations.

    (b) The first number selected can be any one of the 20 options; the second number selected can be any one of 19 options (since the number selected first can't be used twice); the third number selected can be any one of 18 options (since the first and second numbers selected can't be used twice). It follows that there are $\underline{\underline{20 \cdot 19 \cdot 18 = P(20, 3)}}$ different combinations possible if no number can be used twice.

24. There are 20 options for the first entry, 9 options for the second entry, 6 options for the third entry, and 20 options for the last entry. Consequently, there are $\underline{\underline{20^2 \cdot 9 \cdot 6}}$ different ciphers that the CIA operative can send to his Russian KGB counterpart.

25. Since there are seven white keys and we can repeat a note once we've selected it, there are $\underline{\underline{7^8}}$ different eight-note melodies within a single octave that can be written using only the white keys. If the black and white keys need to alternate, there are $\underline{\underline{7^4 \cdot 5^4}}$ different eight-note melodies within a single octave that can be written (there are always seven options when we need to select a white key, and there are always five options when we need to select a black key).

26. We can partition the set of arrangements of no more than three letters formed using the letters of the word NETWORK (without repetition) into the following sets: the set of arrangements using one letter; the set of arrangements using two letters; and the set of arrangements using three letters. The number of elements in the set of arrangements using only one letter is equal to the number of letters in NETWORK, which is $7 = P(7,1)$. To form an arrangement of two letters, we have 7 ways to select the first letter and 6 ways to select the second letter, for a total of $7 \cdot 6 = P(7,2)$ different two-letter arrangements. Finally, for an arrangement using three letters, there are 7 ways to select the first letter, 6 ways to select the second letter, and 5 ways to select the third letter. Thus, there are $7 \cdot 6 \cdot 5 = P(7,3)$ different three-letter arrangements. It follows that there are $P(7,1) + P(7,2) + P(7,3)$ ways to arrange no more than three letters from NETWORK (with no repetitions allowed).

27. We can partition the set of numeric variable names into the following sets: those that begin with a letter and are followed by another letter; those that begin with a letter and are followed by a digit; and those that consist of just a letter. It follows that there are $26^2 + 26 \cdot 10 + 26$ possible numeric variable names. Similarly, we can partition the set of string variable names into the following sets: those that have the form \$ followed by two letters; those that have the form \$ followed by a letter and then followed by a digit; and those that have the form \$ followed by just a letter. Consequently, there are $26^2 + 26 \cdot 10 + 26$ possible string variable names. Thus, there are $2(26^2 + 26 \cdot 10 + 26)$ distinct variable names that this BASIC compiler recognized.

28. This is the same as lining up the nine members in a row, so there are $9! = P(9,9)$ different ways that the Maulers can take the field.

29. The first (left-most) digit must be either a 1 or 2; the second (middle) digit can be any number except for the number chosen for the first digit; and the third (right-most) digit can be any number except for the numbers chosen for the first and second digits. Thus, there are $2 \cdot 6 \cdot 5$ different three-digit numbers formed from the digits 1 through 7, with no digit used more than once, that are less than 289.

30. There are $\binom{4}{3}$ ways to select the three terms in which to take the classes, $\binom{10}{3}$ ways to select the three technical electives, and 3! ways to put the three chosen electives in an order. Thus, there are $\binom{4}{3} \cdot \binom{10}{3} \cdot 3! = \binom{4}{3} \cdot P(10,3)$ different ways in which the engineer can schedule the classes.

31. (a) This is an unordered selection of 15 students taken six at a time; hence, there are $\binom{15}{6}$ ways to select a committee of six from the membership of the council.

    (b) Suppose A and B are the two members with the same major. There are $\binom{13}{4}$ committees of six with both A and B selected. It follows that there are $\binom{15}{6} - \binom{13}{4}$ committees with at most one of A and B selected.

    (c) Suppose A and B are the two members who always insist on serving on committees together. The number of committees of six with neither A nor B selected is equal to $\binom{13}{6}$. On the other hand, the number of committees of six with both A and B selected is equal to $\binom{13}{4}$. It follows that the number of possible committees in this case is $\binom{13}{6} + \binom{13}{4}$.

    (d) i. Since there are $\binom{8}{3}$ ways to select 3 men and $\binom{7}{3}$ ways to select 3 women, there are $\binom{8}{3} \cdot \binom{7}{3}$ ways to select a committee of six containing three men and three women.

ii. The number of committees of six that contain no women is $\binom{8}{6}$. Since the total number of committees possible is $\binom{15}{6}$, there are $\underline{\underline{\binom{15}{6} - \binom{8}{6}}}$ committees of six that contain at least one woman.

(e) There are $\binom{3}{2}$ ways to select two freshmen, $\binom{4}{2}$ ways to select two sophomores, $\binom{3}{2}$ ways to select two juniors, and $\binom{5}{2}$ ways to select two seniors. It follows that there are $\underline{\underline{\binom{3}{2} \cdot \binom{4}{2} \cdot \binom{3}{2} \cdot \binom{5}{2}}}$ committees of eight containing two representatives from each class.

32. (a) This is an unordered selection of ten questions from fourteen questions, so there are $\underline{\underline{\binom{14}{10}}}$ different choices of ten questions.

(b) i. Since there are $\binom{6}{4}$ ways to select four questions that require proof and $\binom{8}{6}$ ways to select six questions that don't require proof, there are $\underline{\underline{\binom{6}{4} \cdot \binom{8}{6}}}$ groups of ten questions containing four that require proof and six that don't.

ii. Since there are only eight questions that don't require proof and we need a group of ten questions, every group of ten questions that we pick has at least one that requires proof. It follows that $\underline{\underline{\binom{14}{10}}}$ groups of ten questions contain at least one that requires proof.

iii. We can partition the groups of ten questions that contain at most three that require proof into the groups that contain exactly two that require proof and those that contain exactly three that require proof (note that every group of ten questions has at least two that require proof). The number of groups that have exactly two that require proof is $\binom{8}{8} \cdot \binom{6}{2}$, and the number of groups with exactly three that require proof is $\binom{8}{7} \cdot \binom{6}{3}$. Consequently, $\underline{\underline{\binom{8}{8} \cdot \binom{6}{2} + \binom{8}{7} \cdot \binom{6}{3}}}$ groups of ten questions contain at most three that require proof.

(c) There are $\binom{12}{8}$ groups of ten questions that include both questions 1 and 2; hence, $\underline{\underline{\binom{14}{10} - \binom{12}{8}}}$ groups of ten questions contain at most one of questions 1 and 2.

(d) There are $\binom{12}{10}$ groups of ten questions that include neither questions 1 nor 2, and $\binom{12}{8}$ groups of ten questions that include both questions 1 and 2. It follows that $\underline{\underline{\binom{12}{10} + \binom{12}{8}}}$ groups of ten questions either include both questions 1 and 2 or exclude both.

33. (a) Since there are 30 members, there are $\underline{\underline{\binom{30}{6}}}$ committees of six that can be formed from the club membership.

(b) The number of committees that contain no men who favored opening the membership to women is $\binom{11}{6}$; the number that contain one man who favored opening membership is $\binom{11}{5} \cdot \binom{19}{1}$; and the number that contain two men who favored opening membership is $\binom{11}{4} \cdot \binom{19}{2}$. Thus, the number of committees of six that contain at least three men who favored opening membership to women is $\underline{\underline{\binom{30}{6} - \left[\binom{11}{6} + \binom{11}{5} \cdot \binom{19}{1} + \binom{11}{4} \cdot \binom{19}{2}\right]}}$

34. (a) There are $\underline{\underline{\binom{40}{4}}}$ different samples that can be chosen.

(b) The number of samples with no defective board is $\binom{37}{4}$, so there are $\underline{\underline{\binom{40}{4} - \binom{37}{4}}}$ samples that contain at least one defective board.

35. We can count this by counting the number of strings of 14 letters, where a letter is either an R or a U and there must be exactly 7 Rs. Since there are $\binom{14}{7}$ ways to place the 7 Rs, and the remaining places must be Us, the rook can follow $\underline{\underline{\binom{14}{7}}}$ different paths from the bottom-left square to the top-right square of the board if all moves are to the right or upward.

36. (a) If the two pilot positions have identical duties, there are $\underline{\underline{\binom{9}{2} \cdot \binom{4}{1}}}$ different crews.

    (b) If the two pilot positions are a pilot and a copilot, there are $\underline{\underline{9 \cdot 8 \cdot \binom{4}{1} = P(9,2) \cdot \binom{4}{1}}}$ different crews.

37. Since the pledge class is to consist of nine members and we want a two-to-one ratio of highly desirable to marginally acceptable candidates, we need to select six highly desirable candidates and three marginally acceptable candidates. Since there are $\binom{10}{6}$ ways to select six highly desirable candidates and $\binom{15}{3}$ ways to select three marginally acceptable candidates, there are $\underline{\underline{\binom{10}{6} \cdot \binom{15}{3}}}$ ways to fill the pledge class.

38. A student needs to be prepared for $\underline{\underline{\binom{7}{5}}}$ possible sets of five essay questions. Order does not matter.

39. (a) The newspaper can choose a set of four interns in $\underline{\underline{\binom{9}{4}}}$ ways.

    (b) If the newspaper must include two men and two women in its set, then there are $\underline{\underline{\binom{5}{2} \cdot \binom{4}{2}}}$ possible sets.

    (c) The number of sets in which everyone is of the same sex is $\binom{5}{4} + \binom{4}{4}$, so the number of sets of four in which not everyone in the set is of the same sex is $\underline{\underline{\binom{9}{4} - \left[\binom{5}{4} + \binom{4}{4}\right]}}$.

40. There are $\underline{\underline{\binom{10}{5}}}$ ways to form two teams of five each.

41. First, we order the 10 ethyls and 5 propyls. We can do this by thinking of forming strings of 15 letters, where each letter is either an E or a P and there must be exactly 10 Es. Thus, there are $\binom{15}{10}$ ways to order the 10 ethyls and 5 propyls. Then there are 16 places were we can put the six methyls (since no two can be next to each other). It follows that $\underline{\underline{\binom{15}{10} \cdot \binom{16}{6}}}$ polymers can be formed if no two of the methyl radicals are to be adjacent.

42. Since two people can only row on the stroke side, we must select two people who can row on either side, of which there are $\binom{3}{2}$ possibilities. Then we need to put the people on each side in an order, so there are $\underline{\underline{\binom{3}{2} \cdot (4!)^2 = \binom{3}{2} \cdot (P(4,4))^2}}$ different ways that the two sides of the boat can be manned.