MATH6 - Introduction to Finite Mathematics

Exam II ANSWERS

May 19, 2007

1. (15 points) Under previous rules, the NCAA men's basketball tournament has 64 teams, paired off to play in 32 first round games. Considering only the first round, how many sets of 32 games are possible among the 64 tournament teams?

Answer:

First, we list the 64 teams in a particular order so that the pairs 1-2, 3-4, etc. represent matchups. There are 64! ways to do this. Then, since each particular game could be listed as A vs. B or as B vs. A, we divide by 2^{32}. Lastly, there is no reason why game 1 and game 2 cannot be switched, and similarly for all 32 games. Since there are 32! ways to list the same set of 32 games, we divide by this number as well, giving the total number of games:

$$\frac{64!}{2^{32}32!}.$$

Alternatively, we could start with any team. Then we must select its opponent from the 63 teams available. Then we set this game aside and choose another, i.e. we select another team and choose its opponent from the 61 remaining. Notice that it does not matter which team we choose first as this only affects the order in which we list the games. We may continue this process until all teams have an opponent, giving

$$63 \cdot 61 \cdot 59 \cdot \cdots \cdot 3 \cdot 1.$$

Note, by writing

$$\begin{aligned} 64! &= (64)(62)(60)\cdots(4)(2) \cdot (63)(61)\cdots(5)(3)(1) \\ &= (2\cdot 32)(2\cdot 31)(2\cdot 30)\cdots(2\cdot 2)(2\cdot 1)\cdot(63)(61)\cdots(5)(3)(1) \\ &= 2^{32}\cdot 32! \cdot (63)(61)\cdots(5)(3)(1), \end{aligned}$$

we see that these two answers give the same value.

(A referee judges a paper's worthiness for publication, usually anonymously)

Referee's report: This paper contains much that is new and much that is true. Unfortunately, that which is true is not new and that which is new is not true. From *Return to Mathematical Circles* by H. Eves.

2. (10 points) Verify the logical equivalence: $(p \to q) \equiv (\neg q \to \neg p)$. What is the name for the statement on the right of this equivalence?

Answer:

We can verify the equivalence by using a truth table:

p	$\neg p$	q	$\neg q$	$p \to q$	$\neg q \to \neg p$
T	F	T	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	T	F	T	T	T

Since the last two columns agree, the two statements are logically equivalent. The statement on the right is called the **contrapositive** of the statement on the left.

3. (15 points) Let $A = \{2n - 1 | n \geq 1 \text{ is an integer}\}$ and $B = \{2n | n \geq 1 \text{ is an integer}\}$.

(a) Is $A \subset B$?

(b) Is $B \subset A$?

(c) Recall $\emptyset = \{\}$. Is $\emptyset \in A$?

(d) Give a representation of the set $A \cap B$ (either by the rule method or by the name of a special set discussed in class; you may use the listing method for half credit).

(e) Give a representation of the set $A \cup B$ (either by the rule method or by the name of a special set discussed in class; you may use the listing method for half credit).

Answer:

(a) No, they are disjoint.

(b) No, they are disjoint.

(c) No, all of the elements of A are integers, not sets.

(d) This is the intersection of the odd positive integers and the even positive integers, namely the empty set, $\emptyset = \{\}$.

(e) This is the union of the positive odd and even integers, which is all positive integers, $\mathbb{N} = \{\text{integers } n | n \geq 1\} = \{1, 2, 3, \ldots\}$.

4. (20 points) The figure below is the sub menu from Tim's Deli. If you want to order a sub, you must select one kind from the meat category, one or no cheese, and any number of the available toppings and condiments. How many different subs may you order?

Meat		Cheese	Toppings		Condiments
Turkey	Roast Beef	Swiss	Lettuce	Tomato	Ketchup
Ham	Italian	Cheddar	Pickles	Green Peppers	Mayo
Veggie		Mozzarella	Jalapenos	Banana Peppers	Mustard
		Provolone	Onions		

Answer:

In ordering your sub, you have five choices of meat (including veggie) and five choices of cheese (including no cheese). Thus there are 25 basic subs without toppings or condiments. To take into account the toppings and condiments, we see that either lettuce is on the sub or it is not. Similarly, either ketchup is on the sub or it is not. So, for each of the 7 toppings and 3 condiments there are 2 possibilities, either they are on the sub or not. Putting this together, the total number of possible subs is given by:

$$5 \cdot 5 \cdot 2^{10} = 25,600$$

5. (25 points) Recall that a standard deck is comprised of 52 cards: 13 denominations (A,2,3,4,5,6,7,8,9,10,J,Q,K) of each of four suits (clubs $\clubsuit$, diamonds $\diamondsuit$, hearts $\heartsuit$, or spades $\spadesuit$). In a game of 5-card stud poker, a flush is a hand of five cards of the same suit. A straight is a hand consisting of five consecutively numbered cards (regardless of suit), where the cards are counted in order A,2,3,4,5,6,7,8,9,10,J,Q,K,A (so an Ace is considered either the lowest or the highest card, whichever the cardholder chooses). A straight flush is both a straight and a flush, i.e. a hand consisting of five consecutive cards from the same suit. Examples of each hand are:

- Straight - $7\diamondsuit$, $8\heartsuit$, $9\heartsuit$, $10\spadesuit$, $J\diamondsuit$
- Flush - $2\clubsuit$, $6\clubsuit$, $7\clubsuit$, $10\clubsuit$, $K\clubsuit$
- Straight Flush - $A\heartsuit$, $2\heartsuit$, $3\heartsuit$, $4\heartsuit$, $5\heartsuit$

Answer the following questions:

(a) What is the probability of a flush?

(b) What is the probability of a straight?

(c) What is the probability of a straight flush?

(d) Are the events straight and flush independent?

Answer:

(a) The number of flush hands is given by $\binom{4}{1}\binom{13}{5}$, so the probability is $P(\text{flush}) = \frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}$.

(b) The number of straight hands is given by choosing a starting point from among the ten possible and then choosing the consecutive cards from the four possibilities for each, i.e. $\binom{10}{1}\binom{4}{1}^5$. Thus the probability is $P(\text{straight}) = \frac{\binom{10}{1}\binom{4}{1}^5}{\binom{52}{5}}$.

(c) The number of straight flushes is given by choosing a suit from the four possible and then a starting point from the ten possible, i.e. $\binom{4}{1}\binom{10}{1} = 40$. The probability of a straight flush is $P(\text{straight} \cap \text{flush}) = \frac{40}{\binom{52}{5}}$.

(d) If these two events are independent, then the product of their probabilities would be the probability of their intersection. However,

$$\frac{P(\text{straight})P(\text{flush})}{P(\text{straight} \cap \text{flush})} = \frac{\frac{\binom{10}{1}\binom{4}{1}^5}{\binom{52}{5}} \frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}}{\frac{40}{\binom{52}{5}}} = \frac{4^5 \binom{13}{5}}{\binom{52}{5}} = \frac{4^2 \cdot 12 \cdot 11}{17 \cdot 5 \cdot 49} \neq 1,$$

so these two events are not independent.

6. (15 points) In sections 1,2, and 3 of Calculus, there are 3, 5, and 2 math majors, while there are 17, 11, and 12 non-math majors, respectively.

(a) If a randomly chosen Calculus student is in section 2, what is the probability that he or she is a math major?

(b) If a randomly chosen Calculus student is a math major, what is the probability that he or she is in section 2?

(c) If a randomly chosen Calculus student is not a math major, what is the probability that he or she is in section 3?

Answer:

(a) There are 5 math majors out of 16 students, so the probability is $\frac{5}{16}$.

(b) Using Bayes' Theorem, we see that the probability is given by:

$$\begin{aligned}
&P(\text{Section } 2|M)\\
&= \frac{P(M|\text{Section } 2)P(\text{Section } 2)}{P(M|\text{Section } 1)P(\text{Section } 1) + P(M|\text{Section } 2)P(\text{Section } 2) + P(M|\text{Section } 3)P(\text{Section } 3)}\\
&= \frac{\frac{5}{16}\frac{16}{50}}{\frac{3}{20}\frac{20}{50} + \frac{5}{16}\frac{16}{50} + \frac{2}{14}\frac{14}{50}}\\
&= \frac{5}{3+5+2}\\
&= \frac{1}{2}.
\end{aligned}$$

(c) Using Bayes' Theorem, we see that the probability is given by:

$$\begin{aligned}
&P(\text{Section } 3|M')\\
&= \frac{P(M'|\text{Section } 3)P(\text{Section } 3)}{P(M'|\text{Section } 1)P(\text{Section } 1) + P(M'|\text{Section } 2)P(\text{Section } 2) + P(M'|\text{Section } 3)P(\text{Section } 3)}\\
&= \frac{\frac{12}{14}\frac{14}{50}}{\frac{17}{20}\frac{20}{50} + \frac{11}{16}\frac{16}{50} + \frac{12}{14}\frac{14}{50}}\\
&= \frac{12}{17+11+12}\\
&= \frac{3}{10}.
\end{aligned}$$

John E. Littlewood (1885-1977): The surprising thing about this paper is that a man who could write it would. From *A Mathematician's Miscellany.*