Math 69
Winter 2009
Monday, January 12

Effectiveness

A set X of expressions is *decidable* if there is an effective procedure that, given any expression ε, will halt after finitely many steps and answer YES in case $\varepsilon \in X$ and answer NO in case $\varepsilon \notin X$.

A set X of expressions is *effectively enumerable* if there is an effective procedure to produce a list of expressions such that, for any expression ε, the expression ε appears on the list (after finitely many steps) iff $\varepsilon \in X$.

A set X of expressions is *semideciable* if there is an effective procedure that, given any expression ε, will halt after finitely many steps and answer YES in case $\varepsilon \in X$, and may run forever or may halt but not answer YES in case $\varepsilon \notin X$.

A function f on expressions is *computable* if there is an effective procedure that, given any expression ε, will halt after finitely many steps and output $f(\varepsilon)$.

Examples:

$\{\alpha \mid \alpha \text{ is a wff}\}$ is decidable. (The parsing algorithm of Section 1.3 determines whether an expression is a wff.)

$\{\alpha \mid \alpha \text{ is a tautology}\}$ is decidable. (The method of truth tables is effective.)

$\{(\alpha, \beta) \mid \alpha \models \beta\}$ is decidable.

If Σ is decidable, so is $\{\vec{\alpha} = (\alpha_1, \alpha_2, \ldots \alpha_n) \mid \vec{\alpha} \text{ is a deduction from } \Sigma\}$.

All the above sets are also effectively enumerable. (There is an effective procedure to list *all* expressions, or pairs of expressions, or finite sequences of expressions. To list a decidable set of expressions X, run the procedure to list the set E of all expressions; after listing an expression in E, test whether it is in X using an effective decision procedure for X, and if it is, then list it in X.)

Do the following problems. Begin with (1) or (2) depending on which group you are assigned to. When you have finished your assigned problem, make sure you are prepared to present your solution to the class before going on to the other problem.

1. Show the following;

A. A set X is effectively enumerable iff it is semidecidable.

B. If Σ is a decidable set of wffs, $\{\alpha \mid \Sigma \vdash \alpha\}$ is effectively enumerable. (Show this without using the Soundness or Completeness Theorems.)

C. Let $f : \mathbb{N} \to \mathbb{N}$ be a computable function whose range is not decidable. (Take it on faith for now that such a function exists.) Let Σ be the set of wffs of the form $\neg\neg\neg\neg \cdots A_{f(n)}$, where $A_{f(n)}$ is preceded by n-many double negations. Show that Σ is decidable but $\{\alpha \mid \Sigma \vdash \alpha\}$ is not decidable.

2. Show the following:

A. A nonempty set X is effectively enumerable iff it is the range of a computable function on $\mathbb{N}$. (The cases where X is finite and X is infinite call for different arguments,)

B. If Σ is a decidable set of wffs, $\{\alpha \mid \Sigma \models \alpha\}$ is effectively enumerable. (Show this without using the Soundness or Completeness Theorems. The Compactness Theorem, however, may be useful.)

C. Suppose that Σ is a decidable set of wffs and for every wff α we have $\Sigma \models \alpha$ or $\Sigma \models (\neg\alpha)$, but not both. Show $\{\alpha \mid \Sigma \models \alpha\}$ is decidable.