Fair Price

Math 5 Crew

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Historical Perspective

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- In summary: Huygens 's Rocks!

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- Suppose you can buy X for b dollars, and sell X for s dollars. Can you sense any conditions that b and s are guaranteed to satisfy?

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- Hence $b \geq s$ or there exist free money! We call this situation an *Arbitrage* opportunities, and the hypothesis that there are no opportunities for Arbitrage is the *No Free Lunch* part of the *Efficient Market Hypothesis.*

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- A *Fair Price* for X would be a price that one could buy or sell X at "among friends". Let us call this Fair Price $E(X)$. Let's try to make sense out of this rather fishy notion.

The Transaction Fee

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- For our Bush bet,

$$f = \frac{65 - 64.3}{2} = .35$$

$$E(X) = 64.65.$$

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- There are many possible factors, but in this model it will depend almost entirely on your access to the market.

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- if we are buying this bet on the market there will be transaction fees associated both to the bets we buy and those we sell. (Consider the bet $X - X$.)

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- In reality we fi nd

$$E(Y) = 83.85.$$

New Bets From Old

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- Among friends, we find that after placing this bet we have

$$12E(Y) - 7E(X) = (12)(83.85) - (7)(64.65) = 553.65$$

dollars worth of debt in our account hence $E(12Y - 7X) = 12E(Y) - 7E(X)$ (by the No Free Lunch and the When There's Cash There's a Way Hypotheses.)

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- the FFMP

$$E(cX + Y + d) = cE(X) + E(Y) + d$$

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- Question: Do we believe that whether or not the Lord of the Rings wins the best picture Oscar will effect Bush's chances of being elected president? If no, we would call X and Y *independent*.

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- I can purchase $E(X)$ shares of Y now. Once Y is determined (the Academy Awards) I will have $YE(X)$ dollars. With my $YE(X)$ dollars (and the above **IF**) I can purchase Y shares of X, in other words XY. So at the end of the day I will have purchased XY for the same price as $YE(X)$, which by the first fundamental mystery has fair price $E(X)E(Y)$.

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- Notice without the big **IF** this really is a mystery to us at this point! Getting rid of the big **IF** would require to fi gure out how to *hedge* a bet in an effi cient market. Later we will (may?) discuss this concept. For now let us just appreciate the mystery!

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- Notice if either Bush or Lord of the Rings loses, then I get nothing. If they both win I get 10000. That this bet is fair tells me that the current belief is that both Bush AND Lord of the Rings winning is a better than even bet.