Fair Price

Math 5 Crew

Department of Mathematics

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- In summary: Huygens 's Rocks!

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- Here's one bet. X is 100 dollars if George Bush becomes president and zero otherwise. Let us call a randomly determined number like X a *Random Variable*.
- Suppose you can buy X for b dollars, and sell X for s dollars. Can you sense any conditions that b and s are guaranteed to satisfy?

The Efficient Market Hypothesis

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- After the election you receive and pay 100 dollars, so you still have $b - s$ dollars worth or debt.
- Hence $b \geq s$ or there exist free money! We call this situation an *Arbitrage* opportunities, and the hypothesis that there are no opportunities for Arbitrage is the *No Free Lunch* part of the *Efficient Market Hypothesis.*

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- As expected, there is no Arbitrage.
- A *Fair Price* for X would be a price that one could buy or sell X at “among friends". Let us call this Fair Price $E(X)$. Let’s try to make sense out of this rather fishy notion.

The Transaction Fee

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- A reasonable notion of Fair Price will satisfy $b = E(X) + f$ and and $s = E(X) - f$. Hence $f = (b - s)/2$.
- For our Bush bet,

$$f = \frac{65 - 64.3}{2} = .35$$

$$E(X) = 64.65.$$

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- How many shares of this Bush Bet bet might you be tempted to buy?
- There are many possible factors, but in this model it will depend almost entirely on your access to the market.

Selling and Buying Bets Among Friends.

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- if we are buying this bet on the market there will be transaction fees associated both to the bets we buy and those we sell. (Consider the bet $X - X$.)

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- For example , let Y be 100 dollars if Lord of the Rings wins the Oscar for best picture and zero other wise. This bet will exist.
- In reality we fi nd

$$E(Y) = 83.85.$$

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- Among friends, we fi nd that after placing this bet we have

$$12E(Y) - 7E(X) = (12)(83.85) - (7)(64.65) = 553.65$$

dollars worth of debt in our account hence $E(12Y - 7X) = 12E(Y) - 7E(X)$ (by the No Free Lunch and the When There's Cash There's a Way Hypotheses.)

First Fundamental Mystery of Probability

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$$E(cX + Y + d) = cE(X) + E(Y) + d$$

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- This is MUCH trickier. By the When There's Cash There's a Way hypothesis this bet will be offered, but what is its Fair Price?
- Question: Do we believe that whether or not the Lord of the Rings wins the best picture Oscar will effect Bush's chances of being elected president? If no, we would call X and Y *independent*.

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- Now **IF** $E(X)$ is not changing between the time we make our bet and the time of the Academy Awards, then...
- I can purchase $E(X)$ shares of Y now. Once Y is determined (the Academy Awards) I will have $YE(X)$ dollars. With my $YE(X)$ dollars (and the above **IF**) I can purchase Y shares of X, in other words XY. So at the end of the day I will have purchased XY for the same price as $YE(X)$, which by the fi rst fundamental mystery has fair price $E(X)E(Y)$.

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- Notice without the big **IF** this really is a mystery to us at this point! Getting rid of the big **IF** would require to fi gure out how to *hedge* a bet in an effi cient market. Later we will (may?) discuss this concept. For now let us just appreciate the mystery!

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- Notice if either Bush or Lord of the Rings loses, then I get nothing. If they both win I get 10000. That this bet is fair tells me that the current belief is that both Bush AND Lord of the Rings winning is a better than even bet.

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- Notice, from this view $P(\text{Bush is next president}) = E(X/100) = 0.6465$ while the $P(\text{Lord of the Rings Win Best Picture}) = 0.8385$

Discussion Question

- Suppose I offer you the random variable Z which has the property that it is one if at least one pair of you mothers share a birthday and zero otherwise.

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- What do you feel is the fair price for this bet?
- In other words, how likely do feel this coincidence is to happen?

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- Two find we need to find a bet which is 1 if they both win and zero otherwise. Notice, $Z = \frac{X}{100}\frac{Y}{100}$ has this property.
- Hence using the FFMP and SFMP $P(E)$ equals

$$E(Z) = E(\frac{X}{100}\frac{Y}{100}) = \frac{1}{10000}E(XY) = \frac{5420.90}{10000} = 0.542.$$

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- Hence using the fi rst and second fundamental mysteries $P($Bush is next president or Lord of the Rings Win Best Picture$)$ equals

$$E(Z) = E(\frac{X}{100} + \frac{Y}{100} - \frac{X}{100}\frac{Y}{100})$$

$$= \frac{E(X)}{100} + \frac{E(Y)}{100} - \frac{E(XY)}{10000}$$

$$= 0.6465 + 0.8385 - 0.5420 = .9430$$

The Addition Rule

- For any events E_1 and E_2

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- and if E_1 and E_2 are independent from the multiplication rule

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A Measure of Variability

- We wish to measure how far away from from our expected value we **expect** to be. A good choice of how to do this is the expect value of $(X - E(X))^2$. In other words, we defi ne the variance to be

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- Notice from the FFMT

$$V(X) = E(X^2) - 2E(X)E(X) + E(X)^2 = E($$

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- Hence it is tempting to take a square-root, and
- we define the *Standard Deviation* as

$$Sd(X) = \sqrt{V(X)}.$$

Variance Of Our Bush bet

- Let us try and find the variance and and standard deviation of our Bush bet, X. We need to find $E(X^2) - E(X)^2$, and in particular $E(X^2)$. But how?

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- Notice $\frac{X}{100}$ has the property that it can only be 1 or 0. Hence $(\frac{X}{100})^2 = \frac{X}{100}$ and

$$V(X) = 100^2 V(X/100) = 10000(E((X/100)^2) - (E($$

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The Fundamental Theorem of Probability

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- In term of standard deviation we have the Pythagorean Theorem

$$sd(X+Y) = \sqrt{sd(X)^2 + Sd(Y)^2}$$

Proof

- $$V(X+Y)=E((X+Y)^2)-(E(X+Y))^2$$

$$=E(X^2)+E(Y^2)+2E(XY)-E(X)^2-E(Y)^2-2E($$

$$=E(X^2)-(E(X))^2+E(Y^2)-(E(Y))^2$$

$$=V(X)+V(Y)$$

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- and as above $V(X) = 2285.378$ and $V(Y) = 1354.18$, hence

$$V(12Y - 7X) = 144(2285.378) + 49(1354.18) = 3954$$

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- By the FFMP

$$E(A_N) = \frac{1}{N}E(\sum_{i=1}^N X_i) = E(X)$$

Controlling Variance: Little Independent Bets

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- By the FFMP and SFMP

$$V(S_N) = \frac{1}{N^2} V(\sum_{i=1}^{N} X_i) = \frac{V(X)}{N}$$

$$Sd(S_N) = \frac{Sd(X)}{\sqrt{N}}$$

- We standardize a random variable X via
$$X^* = \frac{X - E(X)}{Sd(X)}$$

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- We standardize a random variable X via
$$X^* = \frac{X - E(X)}{Sd(X)}$$
- The Central Limit Theorem: for big enough N A_N^* behaves like the standard normal.
- In particular the probability that $|A_N^*| > 4$ is like one in a million (for suffciently large N).

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- Suppose you were offered to buy a share of the average of 100 independent "Bush-Like" bets ($E(X) = 64.65$, $Sd(X) = 47.8$) for a mere 55 dollars!

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- How many shares of this Averaged Bush Bet bet might you be tempted to buy?
- Well the probability that $|A_N^*| < 4$ is less than one in million. Notice then the probability that

$$A_N < E(A_N) - 4Sd(A_N) = 64.65 - \frac{47.8}{\sqrt{100}} = 59.87$$

Discussion

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- Who are they?!